# Étale cohomology and zeta functions

Krijn Reijnders
r0641742

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In this short expository paper we will show the connection between étale cohomology and the zeta functions. We will first show zeta functons, the Weil conjectures and the need for a Weil cohomology theory. We then introduce étale and $\ell$-adic cohomology as a solution to that need. After that we will show the connection between the two and sketch a proof of the Weil conjectures. The three main works that we mainly use are Hartshorne's Algebraic Geometry [9], Milne's online notes, *"Lectures on Étale Cohomology"* [11] and a paper by Milne [12]. Some material is also drawn from Milne's book on étale cohomology [10]. A very clear introduction into the topic by Dieudonné can be found in [4]. Another, more informal and intuitive, introduction can be found in the review of [10] by S. Bloch in [1].

Although a lot of background information on the development of étale cohomology can be found in these works, we will mainly focus on the mathematics in this paper, to make sure that the length of the text does not get out of hand (the length already got out of hand, because the topic was *too* interesting, I apologize).

The author has had the luck of having a preliminary knowledge on the topics of elliptic curves, $p$-adic numbers and algebraic curves. Although knowledge of the topics is not necessary, a reader may find it useful to know chapter IV of Hartshorne's book [9] and chapter II of Serre's book [13].

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# 1 Zeta functions and the Weil Conjectures

We start this section with an introduction of zeta functions for a scheme $X$. We then state the Weil conjectures, and finish with an example of these Weil conjectures.

## 1.1 Zeta functions of varieties

Let $k = \mathbb{F}_q$ be a finite field (with $q$ elements) and let $X$ be a scheme of finite type over $k$ (so one could think of the zero set of a finite family of polynomials over $k$, affine or projective). Pick one algebraic closure and name it $\bar{k}$. Define

$$\bar{X} := X \times_k \bar{k}$$

to be the scheme over $\bar{k}$ (i.e. $\bar{X}$ is obtained from $X$ by making a *base extension* to $\bar{k}$, see Hartshorne II.3 in [9]). We now denote by $N_r$, for $r \in \mathbb{N}_{>0}$, the number of points of $\bar{X}$ whose coordinates lie in $k_r = \mathbb{F}_{q^r}$.

**Definition 1.** We define the *zeta function* of $X$, denoted $Z(X;t)$ or $Z(t)$ when $X$ is clear, by

$$Z(t) = \exp\left(\sum_{r=1}^{\infty} N_r \frac{t^r}{r}\right).$$

We start with two examples.

**Example 2.** $X = \mathbb{P}^n$.
We are familiar enough with $\mathbb{P}^n$ to see that $N_r = |\mathbb{P}^n(\mathbb{F}_{q^r})|$. This is easy to compute:

$$\mathbb{P}^n := (\mathbb{A}^{n+1} - \{0\})/k^\times$$

so with $|k| = q^r$ we get that

$$N_r = |\mathbb{P}^n(\mathbb{F}_{q^r})| = \frac{(q^r)^{n+1} - 1}{q^r - 1} = q^{r(n)} + q^{r(n-1)} + \dots q^r + 1 = \sum_{k=0}^{n} q^{rk}.$$

We now find that

$$\sum_{r=1}^{\infty}\sum_{k=0}^{n} q^{rk}\frac{t^r}{r} = \sum_{k=0}^{n}\sum_{r=1}^{\infty}\frac{(q^k t)^r}{r} = \sum_{k=0}^{n} -\log(1-q^k t).$$

using the identity $\sum_{r=1}^{\infty} \frac{x^r}{r} = -\log(1-r)$. Therefore, the zeta function of $X$ is

$$Z(t) = \exp\left(\sum_{k=0}^{n} -\log(1-q^k t)\right) = \prod_{k=0}^{n} \frac{1}{1-q^k t}.$$

**Example 3.** Let $X$ be a scheme of finite type over $\mathbb{F}_q$ and denote the number of points of $\bar{X}$ whose coordinates lie in $\mathbb{F}_{q^r}$ as $N_r$. Now look at $X \times \mathbb{A}^1$ and denote the number of points of $\overline{X \times \mathbb{A}^1}$ whose coordinates lie in $\mathbb{F}_{q^r}$ as $M_r$. For every point $x$ in $X(\mathbb{F}_{q^r})$, we can pick an $\alpha \in \mathbb{A}^1_{\mathbb{F}_{q^r}}$ to get $x \times \alpha$ in $X \times \mathbb{A}^1(\mathbb{F}_{q^r})$ so we see that $N_r \cdot q^r = M_r$.

This gives us that

$$Z(X \times \mathbb{A}^1; t) = \exp\left(\sum_{r=1}^{\infty} M_r \frac{t^r}{r}\right) = \exp\left(\sum_{r=1}^{\infty} N_r \frac{(qt)^r}{r}\right) = Z(X; qt).$$

## 1.2 The Weil Conjectures

An important observation in these examples is that the zeta function becomes a rational function of $t$, i.e. a fraction $f/g$ where both $f, g$ are polynomials with rational coefficients. This, and other observations, led Weil to the now-famous *Weil conjectures* in [14], which also gives more historical background. We quote:

**Conjecture** (The Weil Conjectures)**.** *Let $X$ be a smooth projective variety of dimension $d$. Then for the zeta function $Z(t)$ we have*

1. *$Z(t)$ is a rational function of $t$.*
2. *$Z(t)$ satisfies the functional equation*

$$Z\left(\frac{1}{q^d t}\right) = \pm q^{d\chi/2} \cdot t^{\chi} \cdot Z(t)$$

*with $\chi$ equal to the Euler-Poincaré characteristic of $X$ (i.e. the intersection number of the diagonal with itself in $X \times X$)*

3. *We can write*

$$Z(t) = \frac{P_1(t)P_3(t)\cdots P_{2d-1}(t)}{P_0(t)P_2(t)\cdots P_{2d}(t)}$$

*with $P_0(t) = 1 - t$, $P_{2d}(t) = 1 - q^d t$, and for $1 \leq r \leq 2d-1$ we have*

$$P_r(t) = \prod_{i=1}^{\beta_r}(1 - \alpha_{r,i}t)$$

*where the $\alpha_{r,i}$ are algebraic integers of absolute value $q^{r/2}$.*

4. *The degrees $\beta_r$ of the polynomials $P_r(t)$ are called the* Betti numbers *of $X$ and*

$$\chi = \sum_r (-1)^r \beta_r.$$

*If $X$ is the reduction by some prime ideal $\mathfrak{p}$ of $K$ of some smooth projective variety $\tilde{X}$ over a number field $K$ then these Betti numbers are the same as the Betti numbers for $\tilde{X}$.*

The first two conjectures can be proven for curves using the Riemann Roch theorem quite easily (although it would take too much space for this text) and the third, called the analogue of the Riemann hypothesis (we show why in the appendix) was proven by Weil himself for curves, but that is the topic of another paper.

For general varieties, proving the Weil conjectures is not easy. The first two conjectures were proven by Dwork [3] using $p$-adic methods. However, Grothendieck, after some suggestions by Serre, worked out étale cohomology to prove these conjectures. He together with Artin (Michael, son of Emil) in [8] and [5] constructed the foundations we will introduce in the next sections. Then he defined $\ell$-adic cohomology, a construction using étale cohomology. With this, Grothendieck proved the first two conjectures and the last one as well. The third result, the analogue of the Riemann hypothesis, is a much deeper result and needs much more work. This was proven by Deligne in the now famous [2].

**Example 4.** For clarity, let us show the Weil conjectures, using $X = \mathbb{P}^n$, where we already know

$$Z(t) = \prod_{k=0}^{n} \frac{1}{1-q^k t}.$$

1. $Z(t)$ is a rational function of $t$. ☑
2. Using the fact that $\chi = n+1$ in this case and that $\sum_{i=0}^{n} q^i = \frac{n(n+1)}{2}$, we get

$$\begin{aligned} Z\left(\frac{1}{q^n t}\right) &= \prod_{k=0}^{n} \frac{1}{1-\frac{1}{q^{n-k}t}} \\ &= \prod_{k=0}^{n} \frac{q^{n-k}t}{q^{n-k}t-1} \\ &= -q^{n(n+1)/2} \cdot t^{(n+1)} \prod_{k=0}^{n} \frac{1}{1-q^{n-k}t} \\ &= -q^{n(n+1)/2} \cdot t^{(n+1)} Z(t). \qquad ☑ \end{aligned}$$

3. We have $P_r(t) = 1 - q^{r/2}t$ for $r$ even, $P_r(t) = 1$ for $r$ odd $(0 \le r \le 2n)$. ☑
4. So $\beta_r = 1$ for $r$ even and $\beta_r = 0$ for $r$ odd $(0 \le r \le 2n)$. So indeed

$$\chi = n+1 = \sum_r (-1)^r \beta_r$$

and these are the same as the Betti numbers for $\mathbb{P}^n(\mathbb{C})$. ☑

## 2 A Weil cohomology theory

To solve the Weil conjectures, Weil speculated that a cohomology theory for nonsingular projective varieties over a base field $k$ was needed. This is a functor that satisfies the following axioms (see [10] for a more in-depth approach). We introduce this theory in this section, show why the Zariski topology is not suitable for such a theory and introduce the étale topology. With this we define étale and $\ell$-adic cohomology, which does suit the theory.

**Definition 5.** Let $k$ be a base field, and $K$ a field of characteristic zero called the *coefficient field*. A *Weil cohomology theory* is a contravariant functor

$$H^* : \{ \text{ non-singular projective varieties over } k\} \to \{ \text{ graded } K\text{-algebras}\}$$

that satisfies the following eight axioms for a non singular projective algebraic variety $X$ of dimension $n$:

1. $H^i(X)$ are finite dimensional $K$-vector spaces.
2. $H^i(X) = 0$ when $i < 0$ or $i > 2n$.
3. $H^{2n}(X) \cong K$.
4. **The Poincaré duality.** There is a non-degenerate pairing $H^i(X) \times H^{2n-i}(X) \to H^{2n}(X) \cong K$.
5. **The Künneth isomorphism.** $H^*(X) \otimes H^*(Y) \xrightarrow{\sim} H^*(X \times Y)$.
6. **A cycle-map.** There is a map $Z^i(X) \to H^{2i}(X)$ from the group of algebraic cycles of codimension $i$, satisfying functoriality, miltiplicativity and normalization.
7. **The Weak Lefschetz axiom.**For a smooth hyperplane section: $j : W \hookrightarrow X$, we get isomorphisms $j^* : H^i(X) \to H^i(W)$ for $i \leq n-2$ and a monomorphism for $i = n-1$.
8. **The Hard Lefschetz axiom.** The Lefschetz operator $L^i : H^{n-i}(X) \to H^{n+i}(X)$ is an isomorphism for $i = 1, \ldots, n$.

If such a Weil cohomology theory exists, it will solve the Weil conjectures, as we will see in section 3. The path is therefore clear: Find a cohomology theory that satisfies the above axioms and use that to prove the Weil conjectures for the zeta functions of varieties.

### 2.1 The Zariski topology

One might start with the cohomology groups we get from the Zariski topology. However, this does not work, as we will show. Informally we could say that the Zariski topology is too coarse, has too few open subsets, to deal with the fine cohomology of varieties in a sensible way.

Let us briefly introduce cohomology of sheaves by using derived functors such that this idea is clear. For a full elaboration on the topic, see Hartshorne, III [9]. For now, we will have to do with just the basics.

Recall that the constant sheaf $\mathcal{F}$ defined by a discrete abelian group $\Lambda$ is the sheaf with $\mathcal{F}(U) = \{f : \pi_0(U) \to \Lambda\}$ where $\pi_0(U)$ are the connected components of $U$. Work by Grothendieck [6] shows that the category of sheaves on $X$ form an abelian category with enough injectives, allowing us to define an *injective sheaf* using derived functors (the same approach as in ch. III of Hartshorne's [9]).

**Definition 6.** A sheaf $\mathcal{I}$ is called *injective* if for any subsheaf $\mathcal{F}' \subset \mathcal{F}$ a homomorphism $\mathcal{F}' \to \mathcal{I}$ can be extended to $\mathcal{F} \to \mathcal{I}$.

Grothendieck also showed that every sheaf can be embedded into an injective sheaf, thus making it possible to define cohomology groups.

**Definition 7.** The *$r$-th cohomology group* $H^r(X, \mathcal{F})$ is equal to the $r$-th cohomology group of the following chain complex of abelian groups

$$\ldots \leftarrow \mathcal{I}^2(X) \leftarrow \mathcal{I}^1(X) \leftarrow \mathcal{I}^0(X)$$

where each $\mathcal{I}^r$ is injective and the following is an exact sequence

$$0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \mathcal{I}^2 \to \ldots$$

called an *injective resolution.* So

$$H^i(X, \mathcal{F}) = \ker(\mathcal{I}^i(X) \to \mathcal{I}^{i+1}(X)) / \operatorname{Im}(\mathcal{I}^{i-1}(X) \to \mathcal{I}^i(X)).$$

One can show that this does not depend on the injective resolution chosen, however we leave all of the proofs for a class on homological algebra.

Although this approach to cohomology of sheaves has a lot of applications in algebraic geometry, it will not help us find a Weil cohomology theory if we stick to the Zariski topology. This is clear from the following theorem.

**Theorem 8** (Grothendieck's Theorem)**.** *If $X$ is an irreducible topological space, then $H^r(X, \mathcal{F}) = 0$ for all constant sheaves $\mathcal{F}$ and all $r > 0$.*

A proof can be found in Hartshorne, III.2.2.7 in [9].

## 2.2 The étale topology

As we saw in the previous section, there was a need for another topology than Zariski's to construct a Weil cohomology theory. Grothendieck provided us with one, when he introduced the étale topology, which leads to the étale cohomology.

Let $X$ and $Y$ be smooth varieties over an algebraically closed field $k$.

**Definition 9.** A regular map $\varphi : Y \to X$ between smooth varieties is said to be *étale* at a point $y \in Y$ if the tangent map $T_y(Y) \to T_{\varphi(y)}X$ is an isomorphism. Then, $\varphi$ is *étale* if $\varphi$ is étale at every point $y \in Y$.

We can then define the category $\mathrm{Et}/X$ with objects étale maps $U \to X$ and for two objects $\varphi_U : U \to X$ and $\varphi_V : V \to X$ an arrow $f : V \to U$ such that $\varphi_V = \varphi_U \circ f$ with all maps étale.

**Definition 10.** The *étale topology* on $X$ is defined by choosing the étale morphisms $U \to X$ as the open sets. A covering of $U$ is then a family of étale morphisms $\varphi_i : U_i \to U$ over $X$ such that $U = \cup \varphi_i(U_i)$.

Again one can define presheaves and sheaves as contravariant functors $\mathcal{F} : \mathrm{Et}/X \to \mathrm{Ab}$ satisfying analogues of the known axioms for presheaves and sheaves of topological spaces. Again one can show that the resulting category of sheaves is abelian and with enough injectives, allowing us to define étale cohomology groups $H^r(X, \mathcal{F})$ using derived functors.

### 2.2.1 Tangent Cones

For singular varieties, our previous definition of being étale at a point does not work, as the tangent space at a singular point holds too little information. Therefore, we must use the *tangent cone*, which we will show for an affine variety $X = \mathrm{Spec}(k[X_1, \ldots, X_n]/\mathfrak{a})$ where $k$ is a base field, algebraically closed, and $\mathfrak{a}$ is an ideal (this will be enough for varieties, as the tangent cone is a local phenomenon).

**Definition 11.** The *tangent cone* at the origin $O$ is defined by the ideal

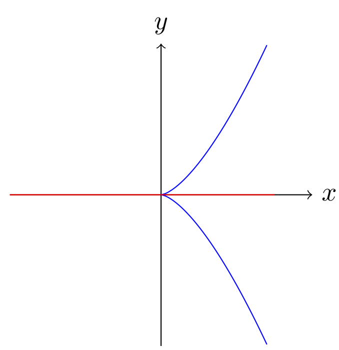
$$\mathfrak{a}_* := \{f_* \mid f \in \mathfrak{a}\}$$

where $f_*$ is the homogeneous part of lowest degree of $f$. The tangent cone $C_O(X)$ is now the affine $k$-scheme

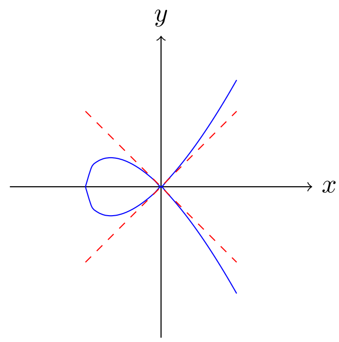
$$C_O(X) := \mathrm{Spec}(k[X_1, \ldots, X_n]/\mathfrak{a}_*)$$

Let us illustrate the concept with two easy examples.

**Example 12** ($X : y^2 = x^3$)**.** We know that $X$ has a singular point at $O$, and we see that the tangent cone is now defined by $y^2 = 0$, so a line at 0 with multiplicity 2, as seen in the following diagram



**Example 13** ($X : y^2 = x^3 + x^2$)**.** We know again that $X$ has a singular point at $O$, and we see that the tangent cone is now defined by $y^2 = x^2$, so $(y - x)(y + x) = 0$, as seen in the following diagram




As is easily seen, the tangent cone is equal to the tangent space in nonsingular points, but captures more information about the singular points. With this we can define what is meant by étale for *all* regular maps of varieties.

**Definition 14.** Let $X$ and $Y$ be varieties over an algebraically closed field $k$. Then a regular map $\varphi : Y \to X$ is called *étale* at $y \in Y$ if the induced map on $C_y(Y) \to C_{\varphi(y)}(X)$ is an isomorphism (or $C_{\varphi(y)}(X) \to C_y(Y)$ if you define a tangent cone as a $k$-algebra instead of a $k$-scheme as some authors do).

Again, we see that this agrees for nonsingular points with the previous definition. So similarly, but now for smooth and non-smooth varieties we define a regular map $\varphi : Y \to X$ to be étale if it is étale at all $y \in Y$. From this we define the category $\mathrm{Et}/X$, the topology, presheaves and sheaves and use homological algebra to define the étale cohomology groups $H^r(X, \mathcal{F})$. With this, we will be able to construct $\ell$-adic cohomology, which will satisfy the axioms of a Weil cohomology theory.

## 2.3 Étale cohomology and $\ell$-adic cohomology as a Weil cohomology theory

We have shown the very basic idea behind the étale cohomology. As it takes a full course to properly explain and understand this cohomology and prove all of the details, we will not do that here. We refer to Milne's notes [11] for those who want to understand this. In the latter sections of the first chapter of these notes, Milne also shows that the étale cohomology indeed satisfies the 8 axioms of a Weil cohomology theory for coefficients $\mathbb{Z}/n\mathbb{Z}$ for $n$ co-prime to $p$, where $p$ is the characteristic of the finite field $\mathbb{F}_q$.

However, for non-torsion coefficients, étale cohomology gives unsatisfactory results. One extra step is needed to also give satisfactory results there to finally get the Weil cohomology theory we need.

**Definition 15.** Let $\ell$ be a prime different from $p$. For a scheme $X$ the *$\ell$-adic cohomology group* $H^i(X, \mathbb{Z}_\ell)$ (here $\mathbb{Z}_\ell$ denotes the $\ell$-adic integers) is the inverse limit of the étale cohomology groups $H^i(X, \mathbb{Z}/\ell^k\mathbb{Z})$, i.e.

$$H^i(X, \mathbb{Z}_\ell) := \lim_{\leftarrow} H^i(X, \mathbb{Z}/\ell^k\mathbb{Z})$$

(One should be careful here, as cohomology does not commute with taking inverse limits, and so this notation may be mistaken for the étale cohomology with coefficients in $\mathbb{Z}_\ell$.)

We can then also tensor with $\mathbb{Q}_\ell$ to remove any torsion subgroup from these cohomology groups, therefore we define

$$H^i(X, \mathbb{Q}_\ell) := H^i(X, \mathbb{Z}_\ell) \otimes \mathbb{Q}_\ell.$$

(Notice again the misleading notation; we are not describing étale cohomology with coefficients in $\mathbb{Q}_\ell$.)

Showing that this cohomology theory is really a suitable Weil cohomology theory was the great work done by Grothendieck and Artin in [8] and [5].

# 3 Proof of the Weil conjectures using étale and $\ell$-adic cohomology

We have now introduced étale and $\ell$-adic cohomology and have accepted that it is a suitable Weil cohomology theory to solve the Weil conjectures with. In this section we will therefore try to sketch the proof of conjecture 1, *rationality*, conjecture 2, *the functional equation*, and conjecture 4, *the Betti numbers*, using étale and $\ell$-adic cohomology. The proof of the analogue of the Riemann hypothesis, conjecture 3, is too deep and complicated to sketch here, but we refer to Deligne's original proof [2], or the books [4], [10] and [11] for those interested. In the end of the section we also work out some examples to show the strength of the Weil conjectures.

## 3.1 The first Weil conjecture

In this section we shall show that the zeta function is indeed a rational function with coefficients in $\mathbb{Q}$. We have defined the zeta function of a variety $X$ as

$$Z(t) = \exp\left(\sum_{r=1}^{\infty} N_r \frac{t^r}{r}\right).$$

Then we see that

$$\log(Z(t)) = \sum_{r=1}^{\infty} N_r \frac{t^r}{r}.$$

This is where $\ell$-adic cohomology comes in. Grothendieck was able to prove that the Lefschetz fixed-point formula holds for $\ell$-adic cohomology in [7] which implies the following relation between the zeta function and the $\ell$-adic cohomology

$$N_r = \sum_{n=0}^{2d} (-1)^n \operatorname{Tr}(F^r \mid H^n(X, \mathbb{Q}_\ell))$$

with $F : X \to X$ the Frobenius map. Now, we use the following lemma from algebra

**Lemma 16.** *Let $T : V \to V$ be an endomorphism of a finite-dimensional vector space $V$, then*

$$\log(\det(1 - Ts \mid V)) = -\sum_{r=1}^{\infty} \operatorname{Tr}(T^r \mid V) \frac{s^r}{r}$$

*Proof.* Define $P_T(s) = \det(1 - Ts \mid V)$ as the characteristic polynomial of $T : V \to V$ then if

$$P_T(s) = \prod_i (1 - \lambda_i s)$$

we know from simple linear algebra that

$$\operatorname{Tr}(T^r \mid V) = \sum_i \lambda_i^r.$$

So we get

$$\begin{aligned} -\log(P_T(s)) &= \log \frac{1}{P_T(s)} \\ &= \log \prod_i \frac{1}{(1 - \lambda_i s)} \\ &= \sum_i \log \frac{1}{(1 - \lambda_i s)} \\ &= \sum_i \sum_{r=1}^{\infty} \frac{(\lambda_i \cdot s)^r}{r} \\ &= \sum_{r=1}^{\infty} \operatorname{Tr}(T^r \mid V) \frac{s^r}{r} \end{aligned}$$

which completes the proof. □

So we see that for $T = F^* : H^n(X, \mathbb{Q}_\ell) \to H^n(X, \mathbb{Q}_\ell)$ and $V = H^n(X, \mathbb{Q}_\ell)$

we would get that

$$\begin{aligned}
\log(Z(X;t)) &= \sum_{r=1}^{\infty} N_r \frac{t^r}{r} \\
&= \sum_{r=1}^{\infty} \sum_{n=0}^{2d} (-1)^n \operatorname{Tr}(F^{*r} \mid H^n(X, \mathbb{Q}_\ell) \frac{t^r}{r} \\
&= \sum_{n=0}^{2d} (-1)^n \sum_{r=1}^{\infty} \operatorname{Tr}(F^{*r} \mid H^n(X, \mathbb{Q}_\ell) \frac{t^r}{r} \\
&= \sum_{n=0}^{2d} (-1)^{n+1} \log(\det(1 - F^* t \mid H^n(X, \mathbb{Q}_\ell)) \\
&= \log\left( \prod_{n=0}^{2d} \det(1 - F^* t \mid H^n(X, \mathbb{Q}_\ell))^{(-1)^{n+1}} \right)
\end{aligned}$$

So we find

$$Z(X;t) = \prod_{n=0}^{2d} \det(1 - F^* t \mid H^n(X, \mathbb{Q}_\ell))^{(-1)^{n+1}}$$

Comparing this to the original conjectures, we find that we have proven

**Theorem 17.**

$$Z(t) = \frac{P_1(t)P_3(t)\cdots P_{2d-1}(t)}{P_0(t)P_2(t)\cdots P_{2d}(t)}$$

*where*

$$P_r(t) = \det(1 - F^* t \mid H^r(X, \mathbb{Q}_\ell)).$$

Indeed, this proves that we can write the zeta function of an appropriate variety $X$ as a rational function, however, the coefficients might now come from $\mathbb{Q}_\ell$. So, for the first Weil conjecture, we would still need that the coefficients really are from $\mathbb{Q}$. Therefore, we will also need the following result

**Theorem 18.**

$$Z(X;t) = \prod_x \frac{1}{1 - t^{\deg x}}$$

*where $x$ runs over the closed points of $X$.*

which we show in the following section (consequence 20).

#### 3.1.1 Rewriting the zeta function

We saw that

$$\log(Z(t)) = \sum_{r=1}^{\infty} N_r \frac{t^r}{r}.$$

Now $N_r$ is the number of points of $X$ in $X(\mathbb{F}_{q^r})$. Let us take a closer look at closed points of $X$. Let $x \in X$ be a closed point, then by definition, the residue field $\kappa(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$. If a closed point $x \in X$ is in $X(\mathbb{F}_{q^r})$ then $\kappa(x)$ is a finite extension of $\mathbb{F}_q$ and so we can define

$$\deg(x) := [\kappa(x) : \mathbb{F}_q].$$

We can then wonder, how many points does $x$ contribute to $N_r$. We call this number $N_r(x)$.

**Lemma 19.**

$$N_r(x) = \begin{cases} \deg x & \textit{if } \deg x | r \\ 0 & \textit{otherwise.} \end{cases}$$

A proof of this can be found on page 156 of [11]. With this we can rewrite the zeta function of $X$ into a more usable form

**Consequence 20.**

$$Z(X;t) = \prod_x \frac{1}{1 - t^{\deg x}}$$

*where $x$ runs over the closed points of $X$.*

*Proof.* We have already used the identity $\sum_{r=1}^{\infty} \frac{t^r}{r} = -\log(1-r)$, so we see now that

$$\begin{aligned}
\log(Z(X;t)) &= \sum_{r=1}^{\infty} N_r \frac{t^r}{r} \\
&= \sum_x \sum_{r=1}^{\infty} N_r(x) \frac{t^r}{r} \\
&= \sum_x \sum_{r=1}^{\infty} \begin{cases} \deg(x) \frac{t^r}{r} & \text{if } \deg x | r \\ 0 & \text{otherwise.} \end{cases} \\
&= \sum_x \sum_{\deg x | r} \frac{\deg(x)}{r} \cdot t^r \\
&= \sum_x \sum_{n=1}^{\infty} \frac{1}{n} t^{\deg x \cdot n} \qquad \text{where } n = r / \deg x \\
&= \sum_x \sum_{n=1}^{\infty} \frac{(t^{\deg x})^n}{n} \\
&= \sum_x \log \frac{1}{1 - t^{\deg x}} \\
&= \log \prod_x \frac{1}{1 - t^{\deg x}}
\end{aligned}$$

which proves the consequence. □

So we have now found 2 ways to write $Z(X;t)$:

1. We can write $Z(X;t)$ as a rational function with coefficients in $\mathbb{Q}_\ell$ (theorem 17).
2. We can write $Z(X;t)$ as $\prod_x \frac{1}{1-t^{\deg x}}$ where $x$ runs over the closed points of $X$ (consequence 20).

The first implies that $Z(X;t) \in \mathbb{Q}_\ell(t)$ and the second implies that $Z(X;t) \in \mathbb{Q}[[t]]$. But then, by Bourbaki (2, Ch. IV  5, exercise 3), which says that

$$\mathbb{Q}[[t]] \cap \mathbb{Q}_\ell(t) = \mathbb{Q}(t)$$

we finally find that $Z(X;t)$ is indeed a rational function with coefficients in $\mathbb{Q}$. We have thus sketched the proof of the first Weil conjecture. We will move on to the fourth Weil conjecture, because we will need it to prove the second Weil conjecture.

## 3.2 The fourth Weil conjecture

Using consequence 20 we can also say something about the fourth Weil conjecture

**Conjecture** (4)**.** *The degrees $\beta_r$ of the polynomials $P_r(t)$ are called the* Betti numbers *of $X$ and*

$$\chi = \sum_r (-1)^r \beta_r.$$

*These Betti numbers are the same as the Betti numbers for $X(\mathbb{C})$.*

Namely, we now know the degrees $\beta_r$ of the polynomials

$$P_r(t) = \det(1 - F^* t \mid H^r(X, \mathbb{Q}_\ell)).$$

So the Betti number is precisely the rank of the $r$-th cohomology group and 0 for $r > 2d$.

We defined $\chi$ as the intersection number of the diagonal with itself in $X \times X$. As the $\ell$-adic cohomology is a Weil cohomology, we get the following theorem

**Theorem 21** (Lefschetz Trace Formula)**.** *Let $\varphi : X \to X$ be a regular map such that $\Gamma_\varphi \cdot \Delta$ is defined, where $\Gamma_\varphi$ is the graph of $\varphi$ and $\Delta$ the diagonal. Then the following holds*

$$(\Gamma_\varphi \cdot \Delta) = \sum_{r=0}^{2d} (-1)^r \operatorname{Tr}(\varphi \mid H^r(X, \mathbb{Q}_\ell))$$

If we apply this formula to $\varphi = \mathrm{Id}$ then we get

$$\chi = (\Gamma_{\mathrm{Id}} \cdot \Delta) = \sum_{r=0}^{2d} (-1)^r \operatorname{Tr}(\mathrm{Id} \mid H^r(X, \mathbb{Q}_\ell)) = \sum_{r=0}^{2d} (-1)^r \beta_r$$

We are left to show that these numbers are indeed the same as the Betti numbers for $X(\mathbb{C})$. This follows also from the fact that the $\ell$-adic cohomology is a Weil conjecture, but requires more work. We refer to chapters 20 and 21 in [11].

## 3.3 The second Weil conjecture

In this section, we will show the second Weil conjecture

**Conjecture** (2). *$Z(t)$ satisfies the functional equation*

$$Z\left(\frac{1}{q^d t}\right) = \pm q^{d\chi/2} \cdot t^{\chi} \cdot Z(t)$$

*with $\chi$ equal to the Euler-Poincaré characteristic of $X$ (i.e. the intersection number of the diagonal with itself in $X \times X$).*

We know already from the previous section that $\chi = \sum_{r=0}^{2d} (-1)^r \beta_r$. As the $\ell$-adic cohomology was proven to be a Weil cohomology theory, we also have that it satisfies axiom 4, the Poincaré duality, i.e. there exists a perfect pairing

$$H^i(X, \mathbb{Q}_\ell) \times H^{2d-i}(X, \mathbb{Q}_\ell) \to H^{2d}(X, \mathbb{Q}_\ell).$$

To this perfect pairing we apply the following lemma (see also Hartshorne C.4.3 in [9]) from linear algebra

**Lemma 22.** *Let $V \times W \to K$ be a perfect pairing of vectorspaces of dimension $r$ over $K$ and let $\lambda \in K$. Then for endomorphisms $\varphi : V \to V$ and $\psi : W \to W$ s.t.*

$$\langle \varphi v, \psi w \rangle = \lambda \langle v, w \rangle$$

*for all $v \in V$ and all $w \in W$, we get*

$$\det(1 - \psi t \mid W) = \frac{(-1)^r \lambda^r t^r}{\det(\varphi \mid V)} \det\left(1 - \frac{\psi}{\lambda t} \mid V\right)$$

*and*

$$\det(\psi \mid W) = \frac{\lambda^r}{\det(\varphi \mid V)}$$

Recall that

$$P_r(t) = \det(1 - F^* t \mid H^r(X, \mathbb{Q}_\ell)).$$

so as a consequence of the last lemma, take $F^* : H^i \to H^i$ and $F^* : H^{2d-i} \to H^{2d-i}$. We use the fact that $F^*$ is compatible with the cup structure and the fact that $F^*$ is multiplication by $q$ in degree 2, we get that $F^*$ is multiplication by $q^d$ in $H^{2d}(X, \mathbb{Q}_\ell)$. So we get that $\lambda = q^d$. Lastly, we know the dimension of $H^i$ is exactly $\beta_i$ from the fourth Weil conjecture. Throwing this al together, we get the two results

$$P_{2d-i}(t) = \frac{(-1)^{\beta_i} q^{d^{\beta_i}} t^{\beta_i}}{\det(F^* \mid H^i(X \mid \mathbb{Q}_\ell))} P_i\left(\frac{1}{q^d t}\right)$$

and

$$\det(F^* \mid H^{2d-i}(X \mid \mathbb{Q}_\ell)) = \frac{q^{d^{\beta_i}}}{\det(F^* \mid H^i(X \mid \mathbb{Q}_\ell))}$$

With this, we can prove the conjecture

**Theorem 23.** *$Z(t)$ satisfies the functional equation*

$$Z\left(\frac{1}{q^d t}\right) = \pm q^{d\chi/2} \cdot t^\chi \cdot Z(t)$$

*with $\chi$ equal to the Euler-Poincaré characteristic of $X$ (i.e. the intersection number of the diagonal with itself in $X \times X$)*

*Proof.* We know from theorem 17 that

$$Z(t) = \frac{P_1(t)P_3(t)\cdots P_{2d-1}(t)}{P_0(t)P_2(t)\cdots P_{2d}(t)}$$

where

$$P_r(t) = \det(1 - F^* t \mid H^r(X, \mathbb{Q}_\ell)).$$

and we have just proven that

$$P_{2d-i}(t) = (-1)^{\beta_i} t^{\beta_i} \det(F^* \mid H^{2d-i}) P_i\left(\frac{1}{q^d t}\right).$$

First notice that

$$\det(F^* \mid H^{2d-i}(X \mid \mathbb{Q}_\ell)) \cdot \det(F^* \mid H^i(X \mid \mathbb{Q}_\ell)) = q^{d\beta_i}$$

so that

$$\begin{aligned}
\frac{\prod_{i \text{ odd}} \det(F^* \mid H^{2d-i}(X \mid \mathbb{Q}_\ell))}{\prod_{i \text{ even}} \det(F^* \mid H^{2d-i}(X \mid \mathbb{Q}_\ell))} &= \frac{1}{\det(F^* \mid H^d(X \mid \mathbb{Q}_\ell))} \frac{\prod_{i \text{ odd}}^{d-1} q^{d\beta_i}}{\prod_{i \text{ even}}^{d-2} q^{d\beta_i}} \\
&= q^{-d\beta_d/2} \cdot q^{d\sum_{i=0}^{d-1}(-1)^{i+1}\beta_i} \\
&= q^{-d\chi/2}
\end{aligned}$$

which finally allows us to compute

$$\begin{aligned}
Z(t) &= \frac{\prod_{i \text{ odd}} P_i(t)}{\prod_{i \text{ even}} P_i(t)} \\
&= (-1)^{\sum(-1)^{i+1}\beta_i} t^{\sum(-1)^{i+1}\beta_i} q^{-d\chi/2} \frac{\prod_{i \text{ odd}} P_{2d-i}\left(\frac{1}{q^d t}\right)}{\prod_{i \text{ even}} P_{2d-i}\left(\frac{1}{q^d t}\right)} \\
&= (-1)^{-\chi} t^{-\chi} q^{-d\chi/2} Z\left(\frac{1}{q^d t}\right)
\end{aligned}$$

This completes the proof of the second Weil conjecture. □

We see that, using Grothendieck's result that $\ell$-adic cohomology is a Weil cohomology theorem, we were able to prove the first, second and fourth Weil conjecture. The proof of the third Weil conjecture is much more difficult (and also much deeper) so we will not bother to sketch a proof here, we refer again to Deligne's proof [2] (or the books [4], [10] and [11] for those interested). We will show in the last section of this paper why it is called the analogue of the Riemann hypothesis. For now, let us work out three examples to show the strength of the Weil conjectures.

## 3.4 Examples using the Weil conjectures

**Example 24.** Let $X$ be a curve (so dimension 1) of genus $g$ over $k$, so $\chi = 2-2g$ and $\beta_1 = 2g$ so that we know that

$$P_1(t) = \prod_{i=1}^{2g}(1-\alpha_i t).$$

As $P_0(t) = 1-t$ and $P_{2d}(t) = P_2(t) = 1-qt$ we get that

$$Z(X;t) = \frac{\prod_{i=1}^{2g}(1-\alpha_i t)}{(1-t)(1-qt)} = \exp\left(\sum_{r=1}^{\infty}(1+q^r-\sum_{i=1}^{2g}\alpha_i^r)\frac{t^r}{r}\right)$$

so we find that

$$N_r = 1+q^r-\sum_{i=1}^{2g}\alpha_i^r$$

for all $r \geq 1$. Hence, we know all $N_r$ if we know $N_1$ up to $N_{2g}$! But we can do even better if we use the functional equation

$$Z\left(\frac{1}{qt}\right) = \pm q^{(1-g)}t^{(2-2g)}Z(t)$$

so that

$$\frac{\prod_{i=1}^{2g}(1-\alpha_i\frac{1}{qt})}{(1-\frac{1}{qt})(1-q\frac{1}{qt})} = \pm q^{(1-g)}t^{(2-2g)}\frac{\prod_{i=1}^{2g}(1-\alpha_i t)}{(1-t)(1-qt)}$$

or rewriting the left side

$$q^{(1-g)}t^{(2-2g)}\frac{\prod_{i=1}^{2g}(\sqrt{q}t-\alpha_i\frac{1}{\sqrt{q}})}{(1-qt)(1-t)} = \pm q^{(1-g)}t^{(2-2g)}\frac{\prod_{i=1}^{2g}(1-\alpha_i t)}{(1-t)(1-qt)}$$

which gives us the comparison

$$\pm\prod_{i=1}^{2g}(\sqrt{q}t-\alpha_i\frac{1}{\sqrt{q}}) = \prod_{i=1}^{2g}(1-\alpha_i t)$$

and by looking at the coefficient of $t^{2g}$ we see that the $\pm$ must be a $+$. So apparently

$$P_1(t) = \prod_{i=1}^{2g} (\sqrt{q}t - \alpha_i \frac{1}{\sqrt{q}}).$$

As we now have two formula describing $\alpha_i$ which must equal each other, we get that we only need to know $N_1$ up to $N_g$ and from there on can figure out $\alpha_i$ for all $1 \leq i \leq 2g$. Hence we will know *all* $N_r$ for $r \geq 1$ from just these $g$ numbers!

**Example 25.** Using the previous example on one of the simpler curves, we let $X$ be an elliptic curve as we know from Hartshorne IV.4 in [9] (although this was not discussed in class, I took a class on elliptic curves in Nijmegen). We know per definition that the genus of an elliptic curve is 1, so we only need to know $N_1$ to determine $Z(t)$. We have that

$$Z(X;t) = \frac{P_1(t)}{(1-t)(1-qt)}$$

and comparing the two formulae for $P_1(t)$ we got, we see

$$(1 - \alpha t)(1 - \beta t) = (\sqrt{q}t - \alpha/\sqrt{q})(\sqrt{q}t - \beta/\sqrt{q})$$

so that

$$Z(X;t) = \frac{1 - (\alpha + \beta)t + qt^2}{(1-t)(1-qt)}$$

with $\alpha\beta = q$ so $|\alpha| = |\beta| = |\sqrt{q}|$. So $N_r = 1 + q^r - \alpha^r - \beta^r$ and more explicitly we have now proven a famous theorem on elliptic curves, namely

**Theorem 26** (Hasse's theorem on elliptic curves)**.**

$$|N_1 - (q+1)| \leq |\alpha| + |\beta| = 2\sqrt{q}$$

**Example 27.** Let us go even more concrete and take an explicit elliptic curve over $\mathbb{F}_2$, say $X : y^2 + y = x^3$. Then we see immediatly that $X(\mathbb{F}_2)$ has 3 points: $(0,0), (0,1)$ and the point at infinity. Therefore, $3 = N_1 = 1 + 2^1 - \alpha^1 - \beta^1$ so $\alpha + \beta = 0$ which gives us the zeta function

$$Z(X;t) = \frac{1 + 2t^2}{(1-t)(1-2t)}.$$

Also, because $\beta = -\alpha$ we get that $\alpha^r + \beta^r = 0$ if $r$ is odd and $\alpha^r + \beta^r = 2\alpha^r$ if $r$ is even. But also $\alpha\beta = 2$ so $\alpha^2 = -2$. Concluding we find

$$|X(\mathbb{F}_{2^r})| = N_r = \begin{cases} 1 + 2^r & \text{if } r \text{ is odd} \\ 1 + 2^r - 2(-2)^{r/2} & \text{if } r \text{ is even} \end{cases}$$

This shows the strength of the Weil conjectures and shows that they give us very *deep* knowledge of varieties but also very *concrete* knowlegde of varieties.

# A Analogue of the Riemann Hypothesis

Here we will show why conjecture 3 is called the *analogue of the Riemann hypothesis*. Let $Y = \mathrm{Spec}(\mathbb{Z})$ then the zeta function of $Y$ is defined as

$$\zeta(Y;s) = \prod_y \frac{1}{1 - \mathbb{N}(y)^{-s}}$$

where $y$ runs over the closed points of $Y$, i.e., the primes. Here $\mathbb{N}(y)$ denotes the order of the residue field at a closed point $y$ of $Y$. We see then that in our case $\mathbb{N}(p) = |\mathbb{Z}/p\mathbb{Z}| = p$ and therefore, $\zeta(Y;s)$ is the original Riemann zeta function. In the same way we can define this zeta function for any scheme of finity type over $\mathrm{Spec}\,\mathbb{Z}$.

Now let $X_0$ be a variety over $\mathbb{F}_q$ of dimension $d$. By the maps

$$\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \hookrightarrow \mathbb{F}_q$$

we get the second map in

$$X_0 \to \mathrm{Spec}\,\mathbb{F}_q \to \mathrm{Spec}\,\mathbb{Z}$$

so we can regard $X_0$ as a scheme of finite type over $\mathrm{Spec}\,\mathbb{Z}$. As we have seen in the previous section in consequence 20

$$Z(X_0;t) = \prod_x \frac{1}{1 - t^{\deg x}}$$

so for $t = q^{-s}$ we get that

$$Z(X_0;q^{-s}) = \prod_x \frac{1}{1 - q^{-s\deg x}} = \prod_x \frac{1}{1 - \mathbb{N}(x)^{-s}} = \zeta(X_0;s).$$

But the third conjecture of Weil states exactly that the zeroes of $Z(X_0;t)$ have absolute value $q^{-1/2}$, $q^{-3/2}$, $q^{-5/2}$, ..., $q^{-(d-1)/2}$ and the poles have absolute value $q^0$, $q^{-1}$, $q^{-2}$, ..., $q^{-d}$ so the zeroes of $Z(X_0;q^{-s}) = \zeta(X_0;s)$ lie on the lines with real part $\frac{1}{2}, \frac{3}{2}, \dots, \frac{d-1}{2}$. We can immediately see the connection with the original Riemann hypothesis.

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