Radboud University Nijmegen

Faculty of Science

Campana curves

ON INTEGRAL POINTS OF BOUNDED HEIGHT

Thesis MSc Mathematics
Algebra and Topology

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Contents

Introduction

This thesis is divided into five chapters. In these five chapters we discuss counting rational, integral and semi-integral points on different types of curves. For rational and integral points, we discuss the major theorems, such as Faltings' theorem and Siegel's theorem. We then introduce semi-integral points and discuss the current conjectures on curves over number fields before looking at (Δ, S)-semi-integral points on curves over function fields.

In the first chapter we discuss rational and integral points on curves over number fields and function fields. To do this properly we will discuss the theory of models of varieties, which we need to define the notion of an integral point on a variety. We will be able to decide on the finiteness (including potential density) of these sets using Faltings' theorem (for rational points) and Siegel's theorem (for integral points).

In the second chapter we introduce Campana orbifolds and Campana curves over number fields. After defining these (using models) and discussing some theory, we move on to the important example of sums of squareful numbers, which can be described by counting (Δ, S)-semi-integral points. The translation of this number-theoretical problem into a Campana orbifold problem was what started my interest in this topic in the first place and has been a very useful tool in explaining my thesis to laymen, such as my mother. We then state the current conjecture on the number of (Δ, S)-semi-integral points of bounded height and discuss the already established results on this conjecture. We also derive how the *abc* conjecture implies the finiteness of the (Δ, S)-semi-integral points for certain curves.

In the third chapter, we extend the discussion of Campana orbifolds to also include function fields as base fields. To do this properly we need to discuss issues that arise over fields of non-zero characteristic, such as isotriviality and separability. We look at how these issues affect the situation over function fields and formulate a new conjecture on the number of (Δ, S)-semi-integral points of bounded height over function fields.

In the fourth and fifth chapter, we treat some approaches to proving the conjecture over function fields and gain some insight in behavior of these (Δ, S)-semi-integral points using data generated in Magma. We will see some small results on this conjecture and discuss an approach using Hom-schemes. We end these chapters and this thesis with a discussion on what lies ahead.

1 Models and integral points

In this section we will explain what a model of a variety over a global field is, and why we need it to define what an *integral point* is. This is necessary as most of the later sections deal with questions about integral points on certain algebro-geometric objects. The theory developed here follows [7] and [1]. We end the section with some important theorems on the number of integral points. We will mainly focus on the theory of points *on curves*. By curve we mean a smooth algebraic curve.

It may be initially unclear why the notion of integral point is nontrivial to define. Let us give an example.

Example 1.1. Let C be the curve over $\mathbb{Q}$ defined by

$$C : X^2 + 4Y^2 = 25$$

in $\mathbb{A}^2_{\mathbb{Q}}$. This clearly has rational points, for example $(3, 2)$ and $(4, 3/2)$. We could also think that $(3, 2)$ is an 'integral point' in a suitable sense. In fact, it has 6 such 'integral' solutions. But now consider

$$C' : X^2 + Y^2 = 25$$

which is isomorphic to C over $\mathbb{Q}$ by $(x, y) \mapsto (x, 2y)$, where we can find 12 'integral' solutions, although the geometric object has not changed!

It can even happen that there are no 'integral' points on one curve, yet another isomorphic curve does have them (for example $C : X^2 + 4Y^2 = 2$ and $C' : X^2 + Y^2 = 2$). We see that indeed, the notion of integral point is not well defined for the isomorphism class of a curve over the base field. This is why we will need models of varieties to accurately describe what we mean by integral points.

1.1 Model of a variety

Let k be a number field and let $X_k \xrightarrow{f} \mathrm{Spec}(k)$ be a variety over k. Then rational points are defined as sections $\mathrm{Spec}(k) \to X_k$ of f. The set of k-rational points is denoted $X_k(k)$. If we have a finite extension k' of k we can also look at the set of k'-rational points on the variety $X_{k'} := X_k \times_{\mathrm{Spec}\, k} k'$ denoted $X_{k'}(k')$. We have

Theorem 1.2 (Faltings)**.** *If C is an algebraic curve over a number field k of genus $g(C) > 1$, then $C(k)$ is finite.*

For curves of genus 0 or 1 we have the following results

Proposition 1.3. *Let C be an algebraic curve over a number field k.*

- *If $g(C) = 0$, then there is an extension k'/k, at most quadratic, such that $C(k')$ is infinite.*

- *If $g(C) = 1$, then there is a finite extension k'/k such that $C(k')$ is infinite.*

We say that the rational points on a variety X are *potentially dense*, if there is a finite extension k'/k such that $X(k')$ is dense in $X_{k'}$ for the Zariski topology. For curves, the infinitude of the set of rational points is equivalent to being dense in the Zariski topology. Thus, in such a case, where the number of points on a curve C becomes infinite after a finite field extension, we say that the rational points on C are *potentially dense.* Thus the finitude of the number of rational points on curves is determined rather nicely by just considering the genus $g(C)$; the number of rational points on a curve with genus larger than 1 is finite, while the rational points on a curve with genus smaller than or equal to 1 are potentially dense. This is also a good moment to introduce the Euler characteristic of a curve C.

Definition 1.4. The Euler characteristic $\chi(C)$ of C is defined to be to

$$\chi(C) := 2 - 2g(C) \in \mathbb{Z}.$$

Note that the Euler characteristic is negative if the genus is larger than 1. This version of the Euler characteristic of a curve is the same as the *topological* Euler characteristic of the associated curve over the complex numbers, and as such we could also define it as the alternating sum of the Betti numbers of C over the complex numbers. This (topological) Euler characteristic $\chi(C)$ is also exactly two times the *holomorphic* Euler characteristic $\chi(C, \mathcal{O}_C)$ defined as in [8, IV.1 Th 1.3].

So we can wrap up these results on curves in the following small table

Table 1.5.

$g(C)$	$\chi(C)$	Number of points
≤ 1	≥ 0	potentially dense
> 1	< 0	finite

We can do this for *global fields* in general. By a global field we mean either an algebraic number field (a finite extension of $\mathbb{Q}$) or a global function field (a finite extension of $\mathbb{F}_q(T)$). These fields can be realised as the fraction fields of Dedekind domains, and that is all we need to develop the theory in this chapter.

So for example for global function fields, the Dedekind domain is the coordinate ring of an affine variety X of transcendence degree 1, for number fields it is the ring $\mathcal{O}_K$. A more complete dictionairy can be found in [14, section 2.6]. For the rest of the section, it might be useful to some readers to think of number fields or function fields instead of global fields.

We will write for a global field $F = k(B)$ for some Dedekind scheme B. By Dedekind scheme, we mean an integral noetherian normal scheme of dimension 1 (for example, the spectrum of a Dedekind domain is a Dedekind scheme). We are interested in rational and integral points on a variety $X_F \xrightarrow{f} \mathrm{Spec}(F)$. A rational point on a variety X_F is again defined as a section of f, i.e. a map

$\mathrm{Spec}(F) \to X_F$. We denote the set of rational points on X_F again as $X_F(F)$. For integral points we do a similar thing. However we will need the notion of a model.

Definition 1.6. Let $X_F \xrightarrow{f} \mathrm{Spec}(F)$ be a variety. A *model* of X_F over B is a scheme $\mathcal{X}_B$ equipped with a flat morphism $\mathcal{X}_B \xrightarrow{g} B$ of finite type and an isomorphism $\mathcal{X}_\eta := \mathcal{X}_B \times_B \eta \xrightarrow{\sim} X_F$ where η is the generic point of B. Equivalently, the following diagram is cartesian:

$$\begin{array}{ccc} X_F & \longrightarrow & \mathcal{X}_B \\ \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle g} \\ \mathrm{Spec}(F) & \xrightarrow{\eta} & B \end{array}$$

A useful property for these models, is *properness*:

Definition 1.7 (The valuative criterion of properness)**.** Let $f : X \to Y$ be a morphism of finite type, with X noetherian. Then f is called *proper* if for every valuation ring R with fraction field K and a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \big\downarrow{\scriptstyle i} & \overset{\vartheta}{\nearrow} & \big\downarrow{\scriptstyle f} \\ \mathrm{Spec}(R) & \longrightarrow & Y \end{array}$$

there is a unique morphism $\mathrm{Spec}(R) \xrightarrow{\vartheta} X$ making the whole diagram commutative.

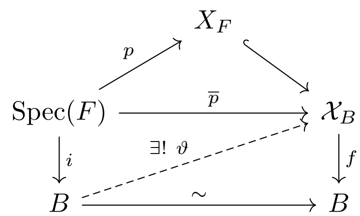
If f is the morphism defininining a variety X_F over $\mathrm{Spec}\, F$ (or a model $\mathcal{X}$ over B), and f is proper, then we call the variety X_F (or the model $\mathcal{X}$) proper.

We will now look at integral points on varieties. As we have shown in the introduction, the notion of an integral point on a variety is not as trivial as it sounds. In fact, we will only be able to associate integral points to a given *model* of a variety.

Recall that $F = k(B)$. We define an integral point of $\mathcal{X}_B$ to be a section $B \to \mathcal{X}$. We denote the set of integral points by $\mathcal{X}(B)$. Often times, we do not want to look at all of B, but at an open subset U of B. A U-integral point will be a section $U \to \mathcal{X}$ of $\mathcal{X}|_U \to U$. With the following lemma we see why the properness of the model is so essential.

Lemma 1.8. *If the model is proper, there is a bijection between the rational points $X_F(F)$ and the integral points $\mathcal{X}_B(B)$.*

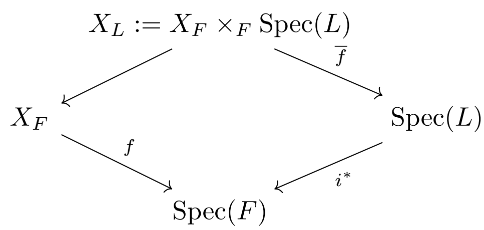
Proof. Let $p \in X_F(F)$. Because $X_F \xrightarrow{\sim} \mathcal{X}_\eta$ we have the following commutative diagram



which gives us a unique section of f named $\vartheta \in \mathcal{X}(B)$ by properness. Any section $\vartheta \in \mathcal{X}(B)$ can of course be restricted to $\mathcal{X}_\eta \xrightarrow{\sim} X_F$ to get a rational point. ☐

Base extension

Let L/F be a finite extension. Then we can extend the variety X_F to a variety over L in the following canonical way (a base change)



Lemma 1.9. *We have*

$$X_F(F) \subset X_L(L).$$

Proof. Let $p \in X_F(F)$. This is a map $p : \mathrm{Spec}(F) \to X_F$. We want a map $P : \mathrm{Spec}(L) \to X_L$. By the universal property of the fiber product, we are looking for maps $p_1 : \mathrm{Spec}(L) \to X_F$ and $p_2 : \mathrm{Spec}(L) \to \mathrm{Spec}(L)$ such that

$$f \circ p_1 = i^* \circ p_2.$$

Let p_1 be the composition of

$$\mathrm{Spec}(L) \xrightarrow{i^*} \mathrm{Spec}(F) \xrightarrow{p} X_F$$

and let $p_2 = \mathrm{Id}$, then $f \circ p_1 = f \circ p \circ i^* = i^* \circ p_2$ because $f \circ p = \mathrm{Id}$. Thus p lifts to a rational point $P \in X_L(L)$. ☐

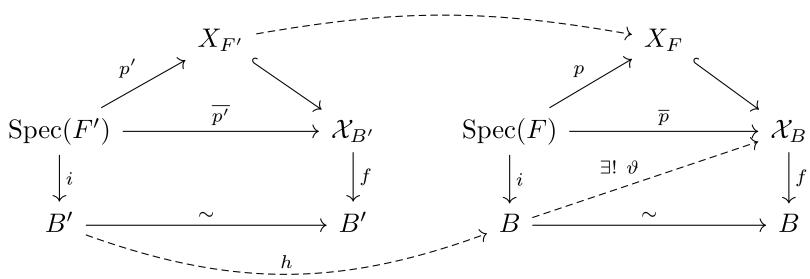
We can also extend the base of a model $\mathcal{X}$ over B.

Lemma 1.10. *Let $B' \to B$ be a finite covering of Dedekind schemes. Let $F' = k(B')$ be the finite extension of F associated to this covering. Then $\mathcal{X}_{B'} := \mathcal{X} \times_B B'$ is a model of $X_{F'}$. If $\mathcal{X}_B$ is a proper model of X_F then again rational points on $X_{F'}$ correspond to integral points on $\mathcal{X}_{B'}$.*

Proof. We get that

$$\begin{aligned}\mathcal{X}_{B'} \times_{B'} \operatorname{Spec}(F') &= \mathcal{X} \times_B B' \times_{B'} \operatorname{Spec}(F') \\ &= \mathcal{X} \times_B \operatorname{Spec}(F') \\ &= \mathcal{X} \times_B \operatorname{Spec}(F) \times_F \operatorname{Spec}(F') \\ &= X_F \times_F \operatorname{Spec}(F') \\ &= X_{F'}.\end{aligned}$$

To show that $\mathcal{X}_{B'}(B') \xrightarrow{\sim} X_{F'}(F')$, we look at the commutative diagrams where we want to find a map $B' \to \mathcal{X}_{B'}$ from the map $p' \in X_{F'}(F')$.



The map $p' : \operatorname{Spec}(F') \to X_{F'}$ induces a map $p : \operatorname{Spec} F \to X_F$, and by properness of $\mathcal{X}_B$ therefore a map $\vartheta : B \to \mathcal{X}_B$. The map $\vartheta' : B' \to \mathcal{X}_{B'}$ is then given by the universal property: we have the map $B' \xrightarrow{h} B \xrightarrow{\vartheta} \mathcal{X}_B$ and the identity map $B' \to B'$, which agree when composed with $\mathcal{X}_B \xrightarrow{f} B$ and $B' \xrightarrow{h} B$, respectively. $\square$

1.2 Integral points on a variety

A crucial result on the number of integral points on a variety is Siegel's theorem. Siegel's theorem is especially concerned with integral points over *number fields*, so let us focus in particular on number fields for this section. In this case we take a variety over a number field K. Let us say we want to define rational and integral points on an affine variety over a number field K. Let S be a finite set of primes of $\mathcal{O}_K$. For an affine curve X over K we look at the smooth compactification $\overline{X}$ (which is unique up to isomorphism and therefore well defined). Let D be the divisor defined by $D := \overline{X} \setminus X$. As K does not have to be algebraically closed, the degree of such a divisor $D = \sum n_p \cdot P$ is defined to be

$$\deg D = \sum n_p \deg P,$$

where the degree of P is defined by $\deg P := \dim_K(\kappa(P))$ with $\kappa(P)$ the residue field. To such a pair (X, D) we can again associate an Euler characteristic.

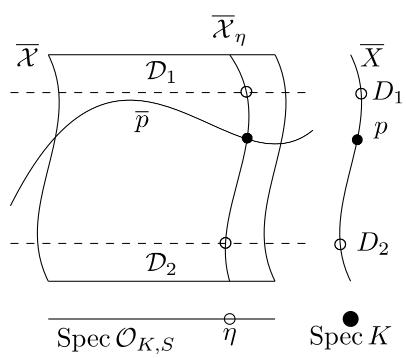
Definition 1.11. The Euler characteristic of the pair (X, D) for a curve X over K and a divisor D is defined to be

$$\chi(X, D) = 2 - 2g(X) - \deg D \in \mathbb{Z}.$$

To define integral points on X we again need an appropriate model. First, take a proper model $\overline{\mathcal{X}}$ of $\overline{X}$ over the ring of S-integers $\mathcal{O}_{K,S}$. Then define $\mathcal{X} := \overline{\mathcal{X}} \setminus \mathcal{D}$ where $\mathcal{D}$ is the Zariski closure of the divisor D to $\overline{\mathcal{X}}$. Now we can define $(\mathcal{D}, S)$-*integral points* on this model.

Definition 1.12. A $(\mathcal{D}, S)$-*integral point* on the model $\mathcal{X} := \overline{\mathcal{X}} \setminus \mathcal{D}$ is a section $\overline{p} : \operatorname{Spec} \mathcal{O}_{K,S} \to \mathcal{X}$ of $\mathcal{X} \to \operatorname{Spec} \mathcal{O}_{K,S}$. We denote the set of all $(\mathcal{D}, S)$-integral points on $\mathcal{X}$ by $\mathcal{X}(\mathcal{O}_{K,S})$.

Such a section is equivalent to a section $\operatorname{Spec} \mathcal{O}_{K,S} \to \overline{\mathcal{X}}$ which does not intersect the divisors above the primes $\mathfrak{p} \notin S$ (we care very little about the behaviour above primes in S and we can even extend S if things are behaving badly, although it must remain finite). See below for an illustration, where we have assumed that D consists of two points, D_1 and D_2.

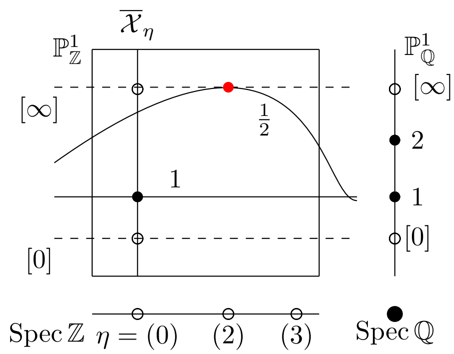


Notice that if the divisor D is empty, we get that integral points and rational points coincide, as we have seen before.

Example 1.13. We can do this even more concretely by looking at a specific affine variety defined over $\mathbb{Q}$

$$X := \operatorname{Spec} \mathbb{Q}\left[t, \frac{1}{t}\right] = \mathbb{G}_{m,\mathbb{Q}}.$$

In this case one would guess that $\operatorname{Spec} \mathbb{Z}[t, \frac{1}{t}]$ makes a good model, and it does! We first compactify X to get $\overline{X} = \mathbb{P}^1_{\mathbb{Q}}$ with $D = [0] + [\infty]$ and thus $\chi(X, D) = 0$. Then a proper model for $\overline{X}$ is $\overline{\mathcal{X}} = \mathbb{P}^1_{\mathbb{Z}}$. The divisors $[0]$ and $[\infty]$ extend in an obvious way and we get that indeed $\mathcal{X} = \overline{\mathcal{X}} \setminus \mathcal{D} = \operatorname{Spec} \mathbb{Z}[t, \frac{1}{t}]$.



The number of integral points depends on the choice of S. If we take $S = \emptyset$ then the $(\mathcal{D}, \emptyset)$-integral points are just $\{\pm 1\}$ as we can see that for example $\frac{1}{2}$ intersects the divisor $[\infty]$ above (2) and in general any $\frac{a}{b}$ with a and b coprime will intersect $[0]$ above (p) when $p \mid a$ and intersect $[\infty]$ above p when $p \mid b$. Thus, a $(\mathcal{D}, \emptyset)$-integral point should have $a = \pm 1$ and $b = \pm 1$. But if we take $S = \{2\}$, then $\frac{1}{2}$ does become a $(\mathcal{D}, S)$-integral point. The $(\mathcal{D}, S)$-integral points are then $\{\pm 2^m \mid m \in \mathbb{Z}\}$, which is an *infinite* set.

In general, the finiteness of the integral points is an interesting question, where we have the following theorems to help.

Proposition 1.14. *Let C be a curve and $g(C) = 0$. Then if $\deg D \leq 2$, and thus $\chi(C, D) \geq 0$, then by a finite extension of K and/or S, any proper model of C has an infinite number of $(\mathcal{D}, S)$-integral points.*

Proof. Let C be a curve with $g(C) = 0$. If necessary extend K to K' such that $C \xrightarrow{\sim} \mathbb{P}^1$. Assume $\deg D = 2$ (the same proof works if $\deg D < 2$), then by the right Möbius transformation we can assume that $D = [0] + [\infty]$. Pick any proper model $\mathcal{C}$ and let $\mathcal{D}$ be the Zariski closure of D.

By properness of $\mathcal{C}$, any rational point p on $\mathbb{P}^1 \setminus D$ will extend to a section $P : \operatorname{Spec} \mathcal{O}_{K'} \to \mathcal{C}$. This section P might still intersect $\mathcal{D}$. However, P will intersect the divisors $[0]$ and $[\infty]$ at most finitely many times, each time above some prime $\mathfrak{p}_i \in \operatorname{Spec} \mathcal{O}_{K'}$. Thus the set of primes $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_k\} \subset \operatorname{Spec} \mathcal{O}_{K'}$ above where P intersects $\mathcal{D}$ is finite. Let $S := \{\mathfrak{p}_1, \ldots, \mathfrak{p}_k\}$, then P is a $(\mathcal{D}, S)$-integral point on $\mathcal{C}$.

Furthermore, if any multiple of P intersects $[0]$ or $[\infty]$, they must intersect above some $\mathfrak{p}_i \in S$. They are therefore integral points. Thus we get an infinite number of $(\mathcal{D}, S)$-integral points on any proper model of C. $\square$

Example 1.15. Let us do an example of a field extension. As in the previous example, take

$$X := \operatorname{Spec} \mathbb{Q}\left[t, \frac{1}{t}\right], \quad \Delta = [0] + [\infty].$$

Instead of enlarging S, we could extend $\mathbb{Q}$ to $\mathbb{Q}(\sqrt{2})$. Then we are looking for sections

$$p : \operatorname{Spec} \mathbb{Z}[\sqrt{2}] \to \mathcal{X}'.$$

Because $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ is a real quadratic extension, $\mathbb{Z}[\sqrt{2}]$ has exactly one fundamental unit, $\varepsilon = 1 + \sqrt{2}$. But then all elements of the *infinite* group of units $\mathbb{Z}[\sqrt{2}]^* = \{\pm\varepsilon^m \mid m \in \mathbb{Z}\}$ are $(\mathcal{D}, S)$-integral points.

Again, as was the case with rational points, we see that these curves have a *potentially dense* set of integral points: If we choose a suitable (finite) extension K'/K or a large enough finite set of finite places S, we get an infinite set of integral points $\mathcal{X}(\mathcal{O}_{K,S})$. And again, as was the case with rational points, this was only possible when $\chi \geq 0$. The other cases, where either $g(C) = 0$ and $\deg D \geq 3$ or where $g(C) > 0$ and $n > 0$, have been settled by Siegel.

Theorem 1.16 (Siegel)**.** *Let X be a curve over a number field K. Then for every choice of $\overline{\mathcal{X}}$ and D (and thus $\mathcal{D}$) such that*

$$\chi(X, D) = 2 - 2g(\overline{X}) - \deg D < 0$$

the set of $(\mathcal{D}, S)$-integral points $\mathcal{X}(\mathcal{O}_{K,S})$ is finite.

Combining these results, we get the following table

Table 1.17.

$\chi(C, D)$	number of integral points
≥ 0	potentially dense
< 0	finite

Comparing this with the previous table (1.5) on rational points we see great similarity. It is possible to prove Siegel's theorem using Faltings's theorem, see for example [1, section 0.3].

2 Campana orbifolds

In this section we introduce *Campana orbifolds.* The Campana orbifold is the same object as the object Campana introduced as 'orbifolds' in [4]. In [1], Abramovich argues that the similarity to original orbifolds is lost quickly, and therefore suggests the name Campana constellation, and more specifically in the case of curves, Campana constellation curves (where we will use *Campana curve*). From this point on, we will strictly use the terminology Campana orbifolds and (more specifically) *Campana curves.*

Campana orbifolds will enable us to describe many number theoretical problems in a different way, which will also help us if we want to describe these problems in a function field setting. We start by defining Campana orbifolds over number fields in section 2.1, we briefly discuss the approach with Campana constellation bases in section 2.2, and follow up with a number theoretical example in 2.3. In the next chapter (chapter 3) we will look at the function field case of Campana orbifolds which will lead to the main question of this thesis (conjecture 3.6).

The leading example of a number theoretical problem where Campana orbifolds come into play is the question on sums of squareful numbers. We call a number $a \in \mathbb{Z}$ *squareful* if for every prime p that divides a, p^2 also divides a. Equivalently, a is squareful if $\mathrm{rad}(a)^2$ divides a where

$$\mathrm{rad}(a) := \prod_{p \ \mid \ a} p.$$

Squareful numbers are also known as *powerful numbers* or 2-*full numbers.* It is not too hard to see that every squareful number can be written uniquely as a product of a square and a squarefree cube. One question we would like to answer is:

How "often" is the sum of two squareful numbers itself squareful?

We are interested in the number of squareful sums of bounded height B. More concretely, we are interested in the growth of $N(B)$ when B goes to infinity, where $N(B)$ is the number of coprime squareful triples $a, b, c \in \mathbb{Z}$ such that $a+b = c$ and $|a|,|b|,|c| < B$. Some examples of these triples are the primitive Pythagorean triples such as $9 + 16 = 25$, or $32 + 49 = 81$ for a non-Pythagorean triple.

We will be counting these triples as points of bounded height on some Campana orbifold. This can be seen as a one-dimensional case of the *Manin conjecture.* The Manin conjecture gives a conjectural approximation for the number of rational points of bounded height on an algebraic variety. When we turn to function fields, we will look at a similar problem of 'squareful' functions, which will be the main part of the thesis as a direct example of the main conjecture.

2.1 Campana curves over number fields

We will need a number field K, its ring of integers $\mathcal{O}_K$ (often with a choice of a finite set of finite primes S), a curve over K which we denote by X and of course a (proper) model $\mathcal{X}$ over $\mathcal{O}_{K,S}$. Finally we need a divisor Δ on X, but instead of integral coefficients, we will also allow rational coefficients δ_p of the form

$$\delta_p = 1 - \frac{1}{m_p}, \quad m_p \in \mathbb{Z}$$

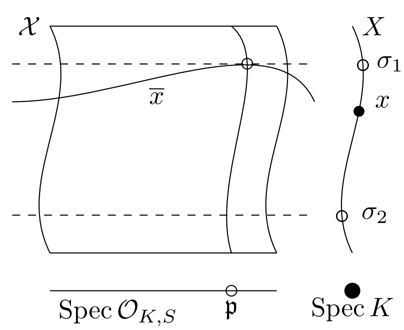
We want the Zariski closure of Δ to $\mathcal{X}$ to be a disjoint union of horizontal divisors over $\mathcal{O}_{K,S}$. As every rational point in the support of Δ extends to an integral point by properness of $\mathcal{X}$, we can achieve this by enlarging S to also contain the primes lying below intersections of these extended divisors if necessary. Write

$$\Delta = \sum \delta_p \sigma_p.$$

We denote the extension of Δ to $\mathcal{X}$ by $\boldsymbol{\Delta}$ We call the pair (X, Δ) a Campana orbifold, and more specifically a Campana curve as X is a curve. We will need a notion of what integral points on such a Campana curve are, with respect to the model $\mathcal{X}$, the divisor $\boldsymbol{\Delta}$ and the finite set of finite primes S. This comes down to the following definition.

Definition 2.1. A K-rational point x on X, considered as an integral point $\overline{x}$ on $\mathcal{X}$ is said to be a (Δ, S)-*semi-integral point* (or *(soft S-)integral point*) on $(\mathcal{X}, \boldsymbol{\Delta})$ if for every $[\sigma]$ of $\boldsymbol{\Delta}$ and any nonzero prime $\mathfrak{p}$ of $\mathcal{O}_{K,S}$ such that $x_\mathfrak{p} = \sigma_\mathfrak{p}$ (i.e. $\overline{x}$ and $[\sigma]$ cross above a prime $\mathfrak{p} \notin S$) the multiplicity of that intersection is larger than or equal to m_p.

This is illustrated in the following image, where $\overline{x}$ intersects $[\sigma_1]$ above $\mathfrak{p}$.



What do we mean, formally, by the multiplicity of such an intersection? Denote the point of intersection above $\mathfrak{p}$ by $x_\mathfrak{p}$ $(= \sigma_\mathfrak{p})$ which lies on both the section $\overline{x}$ and the divisor σ_1. The intersection multiplicity of $\overline{x}$ and σ_1 at the point $x_\mathfrak{p}$ is then defined as $\dim_K \mathcal{O}_{\mathcal{X},x_\mathfrak{p}}/(f,g)$ where f and g are the local equations for $\overline{x}$ and σ_1 respectively, $\mathcal{O}_{\mathcal{X},x_\mathfrak{p}}$ is the local ring at $x_\mathfrak{p}$ of the coordinate ring and the dimension is as a K-vector space (which is non-negative and finite).

In [1, section 2.1] it is explained much better why this is the correct definition of a semi-integral point. What should be clear is that if Δ is a divisor with integral coefficients, then the (Δ, S)-semi-integral points on $(\mathbb{P}^1, \Delta)$ coincide with the (Δ, S)-integral points on $\mathcal{X}$, defined in definition 1.12. One last notion that we need is the Euler characteristic χ for a Campana orbifold.

Definition 2.2. The Euler characteristic χ of the Campana orbifold (X, Δ) with $\Delta = \sum \delta_p \sigma_p$ is defined by

$$\chi(X, \Delta) := \chi(X) - \deg \Delta,$$

where $\deg \Delta := \sum \delta_p \deg \sigma_p$.

By abuse of notation we will often write χ or χ_Δ when the Campana orbifold is deducible. If Δ is a divisor with integral coefficients, $\chi(X, \Delta)$ is again equal to the definition we gave in section 1 (definition 1.11.

2.2 Campana constellation base

The original introduction by Campana came from a different perspective. Rather than choosing a variety X and a divisor Δ, Campana was looking at morphisms $X \xrightarrow{f} Y$ between varieties and tried to deduce knowledge about the rational points on X using the knowledge of certain data associated to f and Y. This gives us the following definition.

Definition 2.3. Let $f : X \to Y$ be a dominant morphism between smooth varieties, with $\dim Y = 1$. Let p be a point of Y and assume the divisor f^*p decomposes on X as

$$f^*p = \sum m_i C_i,$$

with the C_i the irreducible components of the fiber. Then define

$$\delta_p := 1 - \frac{1}{m_p}, \quad m_p := \min m_i,$$

so that we can define the *base divisor* Δ_f on Y as

$$\Delta_f := \sum \delta_p p.$$

We call the pair (Y, Δ_f) the Campana orbifold associated to f.

For example, take an elliptic curve E and the degree 2 map $E \xrightarrow{f} \mathbb{P}^1$ such that f ramifies with degree 2 above 0, 1, some λ and ∞. So we get

$$\Delta_f = \frac{1}{2}[0] + \frac{1}{2}[1] + \frac{1}{2}[\lambda] + \frac{1}{2}[\infty]$$

and the Campana orbifold $(\mathbb{P}^1, \Delta_f)$ with

$$\chi(\mathbb{P}^1, \Delta_f) = 2 - 4 \cdot \frac{1}{2} = 0.$$

The usefulness of these Campana orbifolds induced from $X \xrightarrow{f} Y$ comes from the following key proposition, which is Proposition 2.1.8 from [1].

Proposition 2.4. *Let $X \xrightarrow{f} Y$ be as before with Δ_f the base divisor on Y. If f extends to a good model $\tilde{f} : \mathcal{X} \to \mathcal{Y}$. Then the image of a rational point on X is a (Δ_f, S)-semi-integral point on $(\mathcal{Y}, \Delta_f)$ for a large enough S.*

Although the term 'good model' is never properly defined in [1], Abramovich explained that the following is implied: $\tilde{f} : \mathcal{X} \to \mathcal{Y}$ is flat and smooth away from S and the closure of Δ_f.

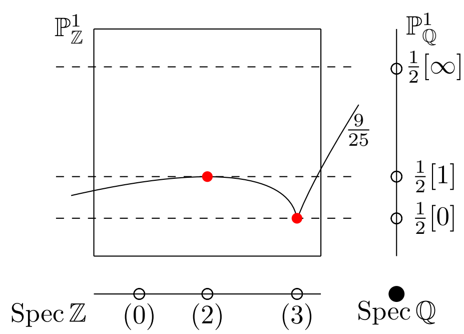
The proof can be found after Proposition 4.3 in [4]. This gives us exactly what Campana was trying to achieve; we can investigate rational points on X using (semi-)integral points on a model of Y. We get

$$X(K) \xrightarrow{f} \{(\Delta_f, S)\text{-semi-integral points on } (\mathcal{Y}, \Delta_f)\},$$

for suitably chosen f.

2.3 Important example of a Campana curve

Let us look at the Campana curve $(\mathbb{P}^1_{\mathbb{Q}}, \Delta)$ over $\mathbb{Q}$ where $\Delta = \frac{1}{2}[0] + \frac{1}{2}[1] + \frac{1}{2}[\infty]$. We will show that (Δ, S)-semi-integral points on this Campana curve correspond to the triples (a, b, c) of squareful numbers such that $a + b = c$. The model we look at will be $\mathbb{P}^1_{\mathbb{Z}}$.



We notice that for example $\overline{p} = \frac{9}{25}$ is now a semi-integral point. Although it intersects the divisor $\frac{1}{2}[\infty]$ above (5), we get that $m_\infty = 2$ is less than or equal to the multiplicity of $\overline{p} \cap \infty$ above (5) which is also 2. Furthermore, it intersects the divisor $\frac{1}{2}[0]$ at (3) where it also has multiplicity $2 \geq m_0$. Lastly, for the divisor $\frac{1}{2}[1]$ we note that $1 - \overline{p} = \frac{16}{25}$ intersects this divisor with multiplicity 4 above (2) which is also larger than $m_1 = 2$. The primes (2), (3) and (5) are the only primes above which $\frac{9}{25}$ intersects Δ. The point $\overline{p}$ corresponds to the triple $9 + 16 = 25$.

In general, we note that if $\overline{n} = \frac{a}{c}$ (where $\frac{a}{c}$ is a reduced fraction) is a semi-integral point on $(\mathbb{P}^1_{\mathbb{Q}}, \Delta)$, then a and c are squareful. The divisor $\frac{1}{2}[1]$ implies that $1 - \overline{n} = \frac{b}{c}$ has b squareful as well and that they obey the rule $a + b = c$. So, semi-integral points on $(\mathbb{P}^1_{\mathbb{Q}}, \Delta)$ correspond to squareful numbers a, b, c such that $a + b = c$.

2.4 Expected number of points

One of the open questions that we are interested in is the growth of the number $n(B)$ of squareful numbers a, b and c such that $a + b = c$ and $a, b, c \leq B$, when $B \to \infty$. In other words, we are interested in the number of (Δ, S)-semi-integral points of bounded height on the Campana oribifold. This is a hard question. The general hypothesis is that when $B \to \infty$ we should get $n(B)/\sqrt{B} \to 1$.

Again, as we saw in Tables 1.5 and 1.17, we would like to count the *potentially dense* number of points. This means that we will sometimes need to enlarge the field K to ensure that at least one such point exists or enlarge the finite set of finite primes S, as we did in Example 1.13. The conjectured number of (Δ, S)-semi-integral points of bounded height on these Campana orbifolds is then the following.

Conjecture 2.5. The number of (Δ, S)-semi-integral points on a Campana orbifold (X, Δ), after a possible finite extension K'/K, with height smaller than B is equal to

Number of points	χ
$\mathcal{O}(B^{\chi})$	$\chi > 0$
$(\log B)^{\mathcal{O}(1)}$	$\chi = 0$
finite	$\chi < 0$

(In the case $\chi < 0$, of course the number of semi-integral points with height smaller than B is finite, but we mean that the total number of (Δ, S)-semi-integral points on the Campana orbifold is finite.)

Example 2.6. In the case where $\Delta = \frac{1}{2}[0] + \frac{1}{2}[1] + \frac{1}{2}[\infty]$ we get

$$\chi(\mathbb{P}^1_{\mathbb{Q}}, \Delta) = 2 - \sum_{i \in \{0,1,\infty\}} \left(1 - \frac{1}{m_i}\right) = 2 - 3 \cdot \frac{1}{2} = \frac{1}{2}$$

which is why we expect a growth of $B^{\frac{1}{2}}$.

We can translate more number theoretical questions into Campana curve questions. See the following table for some conjectures that fit the above conjecture:

Problem	Hypothesis	Δ	χ	Conjecture
How many m-powerful numbers are there?	$B^{1/m}$. (Proven)	$(1-\frac{1}{m})[0]$ $+[\infty]$	$\frac{1}{m}$	$B^{1/m}$
How many squareful triples $a+b=c$?	$\sqrt{B}$	$\frac{1}{2}[0]+\frac{1}{2}[1]$ $+\frac{1}{2}[\infty]$	$\frac{1}{2}$	$\sqrt{B}$
How many consecutive squareful integers are there?	$\mathcal{O}(\log B)$	$\frac{1}{2}[0]+\frac{1}{2}[1]$ $+[\infty]$	0	$(\log B)^{\mathcal{O}(1)}$
Are there three consecutive squareful numbers?	No. (Erdős, 1975)	$\frac{1}{2}[0]+\frac{1}{2}[1]$ $+\frac{1}{2}[2]+[\infty]$	$-\frac{1}{2}$	Finite.

For more information on conjecture 2.5, see for example the presentation [15].

2.5 Established results

We should compare conjecture 2.5 with Tables 1.5 and 1.17. Furthermore, Campana suggested in [4] that by assuming the *abc* conjecture, it should not be too hard to prove that the number of semi-integral points on a Campana curve is finite when $\chi < 0$. This proof was sketched in [1] and finally proven rigorously in [20], which we discuss in section 2.6. An analogue of the *abc* conjecture is proven over function fields, which we use to get a finiteness result in a specific case when we turn to function fields in 4.1.

For the question on sums of squareful numbers, we find some good indications why this conjecture is true in [3]. The authors conjecture a growth of

$$cB^{\frac{1}{2}}(1+o(1))$$

with $c = 2.68...$, which is also their (proven) lower bound and give an upper bound of $\mathcal{O}(B^{3/5}\log^{12} B)$.

2.6 *abc* conjecture implies orbifold Mordell

As promised, we will explain how to prove the finiteness of the number of (Δ, S)-semi-integral points on a Campana orbifold over a number field in the case where $\chi < 0$, assuming the *abc*-conjecture. This result is sometimes called "orbifold Mordell" as it is similar to Faltings' theorem (1.2) which was called the Mordell conjecture until it was proven by Faltings in 1983.

We follow the appendix of [20]. This proof is an adaption of Elkies' proof: "ABC implies Mordell", [6]. Let us first recall some basic definitions and properties of height functions from [9, Part B] that we will need. We write $f = \mathcal{O}(g)$ if there is a positive real number C such that $|f(P)| \leq C \cdot |g(P)|$. For example, a function f is $\mathcal{O}(1)$ if it is bounded by some positive real number C.

Definition 2.7. Let X be a variety over a global field k with an embedding into $\mathbb{P}^n$. Let $P \in X$ and write it as $(x_0 : \ldots : x_n)$ for $x_i \in k$. Then the *height function* H_k and the *logarithmic height function* h_k are defined to be

$$H_k : \mathbb{P}^n \to \mathbb{R}, \quad H_k(P) := \prod_{\mathfrak{p}} \max\{\|x_0\|_{\mathfrak{p}}, \ldots, \|x_n\|_{\mathfrak{p}}\}$$

$$h_k(P) := \log H_k(P) = \sum_{\mathfrak{p}} -n_{v_{\mathfrak{p}}} \min\{v_{\mathfrak{p}}(x_0), \ldots, v_{\mathfrak{p}}(x_n)\}$$

where the product and sum are over all finite and infinite primes. Here, $N_{\mathbb{Q}}^k$ denotes the norm map of k into $\mathbb{Q}$, $v_{\mathfrak{p}}$ denotes the $\mathfrak{p}$-adic valuation of an element of k for some prime $\mathfrak{p}$, $\|x\|_{\mathfrak{p}} := N_{\mathbb{Q}}^k(\mathfrak{p})^{-v_{\mathfrak{p}}(x)}$ and $n_{v_{\mathfrak{p}}}$ is the local degree $[k_{\mathfrak{p}} : \mathbb{Q}_{\mathfrak{p}}]$.

Proposition 2.8. *Let X be a variety over a global field k. To any divisor D on X we can associate a* height function

$$h_D : X(\overline{k}) \to \mathbb{R}$$

with the properties that

1. *if $D \sim D'$ then for any point $P \in X(\overline{k})$*

   $$h_D(P) = h_{D'}(P) + \mathcal{O}(1).$$

   The $\mathcal{O}(1)$ depends on the variety and the divisor, but is independent of the point P.

2. *for a divisor $D + D'$ we get*

   $$h_{D+D'}(P) = h_D(P) + h_{D'}(P) + \mathcal{O}(1).$$

   The $\mathcal{O}(1)$ depends on the variety and the divisor, but is independent of the point P.

3. *if D is* ample*, then for any constant B and finite extension k'/k the set*

   $$\{P \in X(k') \mid h_D(P) \leq B\}$$

   is finite. This is called the Northcott property.

The height function associated to a divisor D is constructed as follows: if D is base point free, take an associated morphism $\phi_D : X \to \mathbb{P}^n$ and let $h_D(P) = h(\phi_D(P))$. If D is not base point free, write $D = D_1 - D_2$ where D_1 and D_2 are base point free, and let $h_D(P) = h_{D_1}(P) - h_{D_2}(P)$. That this is all well defined up to $\mathcal{O}(1)$ can be found in [9, B.4]. This will be all we need to know about height functions.

Let us recall what the *abc* conjecture states. Given $a, b, c \in k^*$ such that $a + b = c$, let the *height* of such a triple be

$$H(a, b, c) = \prod_{\mathfrak{p}} \max\{\|a\|_{\mathfrak{p}}, \|b\|_{\mathfrak{p}}, \|c\|_{\mathfrak{p}}\}$$

with the product taken over all finite and infinite primes of k. Let the *conductor* of such a triple be

$$N(a, b, c) = \prod_{\mathfrak{p} \in I} N(\mathfrak{p})$$

with the product over all finite primes $\mathfrak{p}$ for which

$$\max\{\|a\|_{\mathfrak{p}}, \|b\|_{\mathfrak{p}}, \|c\|_{\mathfrak{p}}\} > \min\{\|a\|_{\mathfrak{p}}, \|b\|_{\mathfrak{p}}, \|c\|_{\mathfrak{p}}\}$$

and $N(\mathfrak{p})$ the absolute norm of the ideal. Then the *abc* conjecture states that, given an $\varepsilon > 0$, there is a constant $C_{\varepsilon,k}$ such that for every triple $(a, b, c) \in \mathbb{Z}^3$ with $a + b = c$ we get that

$$N(a, b, c) \gg C_{\varepsilon,k} H(a, b, c)^{1-\varepsilon} \tag{1}$$

Thus, the goal of this section is to prove:

Theorem 2.9. *Let C be a curve over a number field k and let Δ be a $\mathbb{Q}$-divisor such that the Campana curve (C, Δ) has Euler characteristic $\chi(C, \Delta) < 0$. Then,* assuming the *abc* conjecture*, the set of (Δ, S)-semi-integral points on (C, Δ) is finite, for every finite set of finite primes S.*

Proof. Let $r = -a/b$. Then $N(a, b, c)$ is the product of the absolute norms of all finite primes $\mathfrak{p}$ of k where r, $1/r$ or $1 - r$ has positive valuation, as such a positive valuation of any of these would imply positive $\mathfrak{p}$-adic valuation of one of a, b or c and thus a smaller $\mathfrak{p}$-adic norm of one of a, b or c. Thus we can split $N(r)$ accordingly into

$$N(r) = N_0(r)N_1(r)N_\infty(r)$$

where

$$N_0(r) := \prod_{v_{\mathfrak{p}}(r)>0} N_{\mathbb{Q}}^k(\mathfrak{p})$$

and $N_1(r)$ and $N_\infty(r)$ are defined in a similar way.

Now let $f : C \to \mathbb{P}^1_k$ be a *Belyi map* (this is where this approach over function fields would become impossible), i.e. a map only ramified above 0, 1 and ∞ in such a way that the support of Δ is contained in $f^{-1}(\{0,1,\infty\})$. We define a new rational divisor D_0 on C by

$$D_0 := \sum_{f(\delta)=0} \alpha_\delta \delta.$$

Here the sum runs over the prime divisors δ of C. We set $\alpha_\delta := \frac{1}{m_j}$ if $\delta = \Delta_j$ for some j and $\alpha_\delta = 1$ otherwise. In the same way we define D_1 and D_∞. Then let $D := D_0 + D_1 + D_\infty$. As the support of Δ is fully contained in $f^{-1}(\{0,1,\infty\})$ we get

$$\deg D = 3\deg f - \sum_{f(\delta)\in\{0,1,\infty\}} b_f(\delta)\deg\delta - \sum\left(1-\frac{1}{m_j}\right)\deg\Delta_j,$$

where $b_f(\delta)$ is the branch number of f at δ. Using Riemann-Hurwitz, we get

$$2g(C) - 2 = -2\cdot\deg f + \sum_{f(\delta)\in\{0,1,\infty\}} b_f(\delta)\deg\delta$$

and these two equations together give

$$\begin{aligned}\deg D &= \deg f - 2g(C) + 2 - \sum\left(1-\frac{1}{m_j}\right)\deg\Delta_j \\ &= \deg f - \chi(C,\Delta) \\ &< \deg f\end{aligned}$$

where we use the assumption that $\chi(C,\Delta) < 0$. Let now $p \in C(k)\backslash f^{-1}(\{0,1,\infty\})$. We will show that having an infinite set of (Δ, S)-semi-integral points on (C,Δ) contradicts the *abc* conjecture 4.3 which completes the proof.

Claim 1. We have (similarly for N_1 and N_∞)

$$\log N_0(f(p)) < \frac{\deg D_0}{\deg f}\log H(f(p)) + \mathcal{O}\left(\sqrt{\log H(f(p))}+1\right).$$

Here, the $\mathcal{O}$ term is effective and depends on C but not on p.

Proving the claim will be sufficient, as the additivity of the logarithm, the definitions of N_0, N_1, and N_∞, and the fact that $\deg D = \deg D_0 + \deg D_1 + \deg D_\infty$ will give us

$$\log N(f(p)) < \frac{\deg D}{\deg f}\log H(f(p)) + \mathcal{O}\left(\sqrt{\log H(f(p))}+1\right).$$

Then $f(p)$ would contradict the log version of the *abc* conjecture

$$\log N(f(p)) \gg (1-\varepsilon)\log H(f(p))$$

as soon as

$$1-\varepsilon < \frac{\deg D}{\deg f} < 1,$$

and $H(f(p))$ is sufficiently large (for all but finitely many p).

Proof of Claim 1.

The proof of Claim 1 follows the approach of Elkies' proof: "ABC implies Mordell", [6]. For a divisor c of C, we will denote the associated *height function* (as in prop. 2.8) with $h_c : C(\overline{K}) \to \mathbb{R}$, which is unique up to $\mathcal{O}(1)$ as we have seen. Let

$$\overline{D} := \sum_{f(\delta)=0} n_\delta \delta$$

be the zero divisor of f, which implies that

$$\deg \overline{D} = \sum_{f(\delta)=0} n_\delta \deg \delta = \deg f.$$

By the properties of height functions (2.8), we get that

$$h_{\overline{D}}(p) = \sum_{f(\delta)=0} n_\delta h_\delta(p) + \mathcal{O}(1)$$

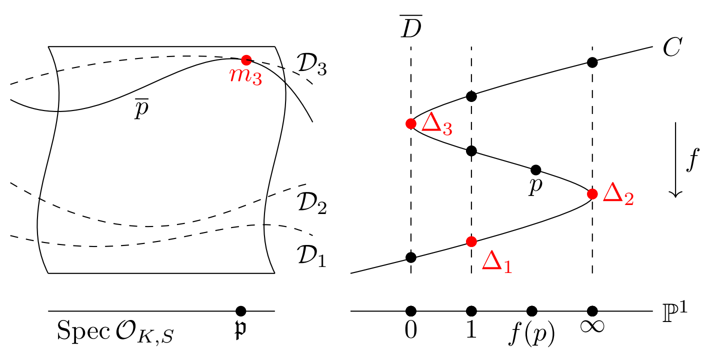
and $h_{\overline{D}}(p) = h(f(p)) + \mathcal{O}(1)$. Thus for $f(p) \in \mathbb{P}^1$ we see that

$$\log H(f(p)) = h(f(p)) = h_{\overline{D}}(p) + \mathcal{O}(1) = \sum_{f(\delta)=0} n_\delta h_\delta(p) + \mathcal{O}(1). \tag{2}$$

Now, recall that

$$N_0(f(p)) = \prod_{v_\mathfrak{p}(f(p))>0} N_\mathbb{Q}^K(\mathfrak{p}).$$

Let $\mathfrak{p}$ be a prime of K outside the set S with $v_\mathfrak{p}(f(p)) > 0$. The contribution to $\log N_0(f(p))$ is then exactly $\log N_\mathbb{Q}^K(\mathfrak{p})$. But $\mathfrak{p}$ also contributes to $H(f(p))$ so by (2), for such a $\mathfrak{p}$, we must have some $\delta \in f^{-1}(0)$ so that $\mathfrak{p}$ also contributes to $h_\delta(p)$. Per the definition of an orbifold rational point (2.1), if $\delta = \Delta_j$ for some $1 \le j \le N$, this contribution to $h_{\Delta_j}(p)$ must be at least $m_j \log N_\mathbb{Q}^K(\mathfrak{p})$.



Therefore, if we sum over all these primes $\mathfrak{p}$ we get that

$$\begin{aligned}\log N_0(f(p)) &= \sum_{v_{\mathfrak{p}}(f(p))>0} \log N_Q^K(\mathfrak{p}) \\ &= \sum_{v_{\mathfrak{p}}(f(p))>0} \frac{m_j}{m_j} \log N_Q^K(\mathfrak{p}) \\ &< \sum_{f(\delta)=0} \alpha_\delta h_\delta(p) + \mathcal{O}(1) \\ &< h_{D_0}(p) + \mathcal{O}(1)\end{aligned}$$

where we recall that $\alpha_\delta = \frac{1}{m_j}$ if $\delta = \Delta_j$ and $\alpha_\delta = 1$ else.

We are left with having to compare $h_{D_0}(p)$ to $h_{\overline{D}}(p)$. To do this, we define a new divisor D' of degree zero on C

$$D' := \deg \overline{D} \cdot D_0 - \deg D_0 \cdot \overline{D}.$$

Thus

$$h_{D'}(p) = \deg \overline{D} \cdot h_{D_0}(p) - \deg D_0 \cdot h_{\overline{D}} + \mathcal{O}(1)$$

implies

$$\frac{h_{D'}(p)}{\deg \overline{D}} = h_{D_0}(p) - \frac{\deg D_0}{\deg \overline{D}} \cdot h_{\overline{D}} + \mathcal{O}(1).$$

Recall that $\deg \overline{D} = \deg f$, then this comes down to

$$h_{D_0}(p) = \frac{\deg D_0}{\deg f} \cdot h_{\overline{D}} + \frac{h_{D'}(p)}{\deg f} + \mathcal{O}(1).$$

As D' has degree zero, we get by a theorem of Néron [18, p.45] that

$$h_{D'}(p) = \mathcal{O}\left(\sqrt{\log H(f(p))} + 1\right).$$

Concluding, we get

$$\begin{aligned}\log N_0(f(p)) &< h_{D_0}(p) + \mathcal{O}(1) \\ &= \frac{\deg D_0}{\deg f} \cdot h_{\overline{D}} + \mathcal{O}\left(\sqrt{\log H(f(p))} + 1\right) \\ &= \frac{\deg D}{\deg f} \log H(f(p)) + \mathcal{O}\left(\sqrt{\log H(f(p))} + 1\right)\end{aligned}$$

which proves the claim. □

3 Campana orbifolds over function fields

In this section we will translate conjecture 2.5 into a problem over function fields. We hope that, as is sometimes the case, the problem is easier over function fields. In particular we will try to answer the analogue of the question on sums of squareful numbers, as it is one of the interesting examples. Although these number theoretical problems are often easier to answer in their function field version, some difficulties arise. We will mostly deal with function fields of positive characteristic, although there exist similar results for function fields of characteristic zero.

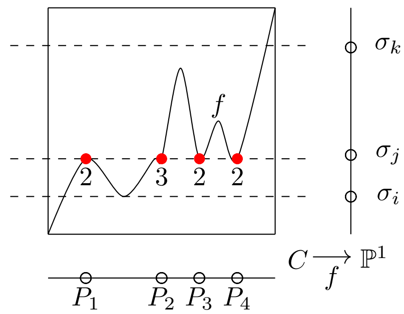
Let us start with a small recap and sketch of the situation. In the previous chapter we explained what Campana orbifolds over number fields were (and in this section we will do the same over function fields). Then we gave the example of the sums of squareful numbers (and the same example will be useful over function fields) and we finished with the conjecture on the number of points and the known results about that conjecture. We will have to modify that conjecture to get a new conjecture over function fields. To do this we will first look at some of the known results that relate to the number of points on Campana curves. We will also look at some of the difficulties that arise with isotriviality (see section 3.1) and inseparability (see section 3.2). This will enable us to formulate a precise conjecture in section 3.3.

Let us, as an example, look at the case where $X = \mathbb{P}^1_k$ for a finite field k with $\operatorname{char} k = p$. For a divisor Δ we will write

$$\Delta = \sum \left(1 - \frac{1}{m_i}\right) [\sigma_i]$$

with $\sigma_i \in \mathbb{P}^1$. Specifically we are interested in the 'squareful' divisor $\Delta = \frac{1}{2}[0] + \frac{1}{2}[1] + \frac{1}{2}[\infty]$. What do (Δ, S)-semi-integral points on this Campana curve $(\mathbb{P}^1, \Delta)$ look like concretely, for a finite set of finite primes S?

As per definition, we are *always* looking at a model $\mathcal{X}$ and embeddings of a curve C into $\mathcal{X}$ as a semi-integral point. When we project these (Δ, S)-semi-integral points onto $\mathbb{P}^1$, we find that we are looking at maps $f : C \to \mathbb{P}^1$ such that above the points σ_i, f ramifies with ramification index larger than or equal to m_i. For example, the f in the image below intersects $[\sigma_j]$ on all 4 points with multiplicity larger than or equal to 2 and equivalently ramifies everywhere above σ_j. It would therefore be a (Δ, S)-semi-integral point on the Campana curve $(\mathbb{P}^1, \frac{1}{2}[\sigma_j])$, but not on $(\mathbb{P}^1, \frac{2}{3}[\sigma_j])$.



We will denote the scheme of such maps $f : C \to \mathbb{P}^1$ with $\mathrm{Hom}(C, \mathbb{P}^1)$ and in particular those of degree d with $\mathrm{Hom}^d(C, \mathbb{P}^1)$, which is a connected component of $\mathrm{Hom}(C, \mathbb{P}^1)$. The (Δ, S)-semi-integral points, i.e. those f with the correct intersection behavior with the divisors, are denoted $\mathrm{Orb}_\Delta(C, \mathbb{P}^1)$ (and $\mathrm{Orb}^d_\Delta(C, \mathbb{P}^1)$ respectively). We will look at these objects more closely in section 5.

3.1 Isotriviality

Because we are dealing with varieties over global function fields, the topic of *isotriviality* is of influence on the conjecture and the results. As in section 1.1, we have a function field F of transcendence degree 1 over a finite field k with $\mathrm{char}\, k > 0$. What is isotriviality and why does it matter for the counting of rational and integral points? Let us first look at the following example:

Example 3.1. Let $F = k(t)$ and let C be a curve over F defined by

$$Y^2 = X^3 + t.$$

Now consider the extension $L := F[t^{1/2}, t^{1/3}]$, then by a change of variables

$$X \mapsto t^{1/3} X', \quad Y \mapsto t^{1/2} Y'$$

we get a curve defined over k

$$C' : \; Y'^2 = X'^3 + 1.$$

Now, C is isomorphic to C' over L by the above change of variables, however C is not isomorphic to C' over F!

In some sense, we see that the curve C defined over a parameter t is in fact isomorphic to the same curve C' for every value of that parameter. We call this phenomenon *isotriviality.* Formally:

Definition 3.2. Let X_F be a variety over F. We say that X_F is *isotrivial* if there exists a variety X_k over k and a finite extension L/F such that

$$X_F \times_F L = X_L \xrightarrow{\sim} X_k \times_{\mathrm{Spec}\, k} \mathrm{Spec}(L).$$

In such a case, every fiber above an element of the base curve will be isomorphic to the fiber over the generic point. This would also give us a number of sections that are constant and thus would automatically be (semi-)integral points. This increases the difficulty in counting the (semi-)integral points.

Luckily, we have the following theorem from [12], which could be seen as an analogue to Falting's theorem (1.2) over function fields of characteristic $p \neq 0$.

Theorem 3.3. *Let C be a curve of genus greater than 2 over a function field F of transcendence degree 1 over a (not necessarily finite) field k of characteristic $p \neq 0$. There are three possibilities;*

- *C is not isotrivial. Then the set $C(F)$ is finite.*
- *C is isotrivial; C is isomorphic to some curve C' defined over k and the field k is infinite: Then the set $C(F)$ is infinite if and only if $C'(k)$ is infinite. Denote by ϕ the isomorphism $C \xrightarrow{\sim} C'$. Then $C(F) \setminus \phi^{-1}C'(k)$ is finite.*
- *C is isotrivial; C is isomorphic to some curve C' defined over $\mathbb{F}_q$ for some q contained in k: Then there is a (Galois) extension L/F and an isomorphism $\phi : C_L \xrightarrow{\sim} C'_L$. The type of extension L/F decides the behavior of rational points:*
    - *If $L = F \cdot k'$ (with $k' = \overline{k} \cap L$) then every point of $C(F)$ can be written as $\phi^{-1}(x)$ for some x in $C'(L)$.*
    - *Else, there is a finite set of points x_i in $C'(L)$ such that every point of $C(F)$ can be written as $\phi^{-1}(f^n(x_i))$ where f is the Frobenius morphism $x \mapsto x^q$ of C'.*

We are mainly focused on the case where the field k is indeed finite and thus will mainly need the first and third part of the theorem. Note that this solves the question on rational points on curves of genus greater than 2 over function fields of characteristic $p \neq 0$: if the curve is not isotrivial, we have a finite number of rational points. If it is isotrivial, then we can find a finite set of points and get all rational points using the Frobenius morphism.

We see that the Frobenius morphism strongly influences the number of rational and integral points. In the next section, we explain this in more detail.

Our main interest lies in the case where $\chi(C, \Delta)$ is larger than 0, which is only possible if we are looking at a Campana curve (C, Δ) where Δ is supported at at most three points and $g(C) = 0$. Let us assume that Δ is supported at three points (the arguments also holds for zero, one, or two points) and $C \xrightarrow{\sim} \mathbb{P}^1$. By a Möbius transformation we can always ensure that Δ is supported at 0, 1 and ∞. In such a case, $(\mathbb{P}^1, \Delta)$ is always isotrivial:

For any proper model of $(\mathbb{P}^1, \Delta)$, we get an extension from 0,1 and ∞ to horizontal divisors [0], [1] and [∞]. Every fiber above a point of C is then a $\mathbb{P}^1$ with three points removed and therefore isomorphic to $\mathbb{P}^1$ with 0, 1 and ∞ removed. Thus every fiber is isomorphic, which means that $(\mathbb{P}^1, \Delta)$ is isotrivial.

Therefore, anytime we are looking at the case $\chi(\mathbb{P}^1, \Delta) > 0$ (which is the main focus in later chapters), we should remind ourselves that such a Campana curve is isotrivial.

3.2 Separability issues

As we will be dealing with isotrivial curves and function fields over a finite field $k = \mathbb{F}_q$ with $\operatorname{char} k = p$, we must also deal with issues regarding the separability of the maps $f : C \to \mathbb{P}^1$. Denote by F the Frobenius morphism $x \to x^p$. For example, in the case where the Campana curve is $(\mathbb{P}^1, \Delta)$ and $\chi(\mathbb{P}^1, \Delta) > 0$, the Campana curve is isotrivial. We will see that for any f which lies on $(\mathbb{P}^1, \Delta)$ of degree d, the map $F(f)$ lies on $(\mathbb{P}^1, \Delta)$ and is of degree dp. Therefore, if such an isotrivial Campana curve has at least one semi-integral point, we must immediately get an infinite number of semi-integral points by repeatedly applying the Frobenius morphism. We will see this re-appear when we are working with data in section 4.4. Even worse, applying F often enough, say n times, would map *any* f to a semi-integral point on the Campana curve, giving a map

$$\mathrm{Hom}^d(C, \mathbb{P}^1) \hookrightarrow \mathrm{Orb}^{d'}(C, \mathbb{P}^1) \quad \text{by} \quad f \mapsto f^{np} \quad \text{where} \quad d' = npd.$$

Definition 3.4. Let (C, Δ) be an isotrivial curve; C is isomorphic to C' defined over k by $\phi : C \times_F L \xrightarrow{\sim} C' \times_k L$ over some finite extension L/F. We call a (Δ, S)-semi-integral point f of (C, Δ) a *Frobenius orbifold point* if it can be written as $\phi^{-1}F(f')$ for some $f' \in C'(L)$, where F is the Frobenius morphism.

Counting these Frobenius orbifold points would thus ruin every chance we had at a conjecture for function fields such as Conjecture 2.5. We therefore suggest to not include these Frobenius orbifold points into the count of points, and only count the (Δ, S)-semi-integral points f such that there is no f' such that $f = F(f')$. Let us look for example what happens in the squareful case.

Example 3.5. Very concretely, it is clear why for $\mathbb{P}^1 \to \mathbb{P}^1$ over $\mathbb{F}_5$ with

$$\Delta = \frac{1}{2}[0] + \frac{1}{2}[1] + \frac{1}{2}[\infty]$$

we get Frobenius orbifold points on $(\mathbb{P}^1, \Delta)$, on any model, of the form

$$f(X) = \frac{(X-\alpha)^5}{(X-\beta)^5} = \frac{X^5 - \alpha}{X^5 - \beta} = F(f'), \quad \text{where } f'(X) = \frac{X-\alpha}{X-\beta}, \quad \alpha, \beta \in \mathbb{F}_5.$$

These points $f \in \mathrm{Hom}^5(\mathbb{P}^1, \mathbb{P}^1)$ are Frobenius orbifold points on the Campana curve $(\mathbb{P}^1, \Delta)$, and so are the Frobenius orbifold points $F^n(f')$ for any integer n greater than zero.

In a more general sense, for $C \to \mathbb{P}^1$ over $\mathbb{F}_q$ of characteristic $p > 0$, any (Δ, S)-semi-integral point f' of degree d can be mapped to a degree $d' = nd$ Frobenius orbifold point $f = F^n(f')$ by the Frobenius automorphism. In section

5 we will be able to calculate the dimension of $\mathrm{Hom}^d(C, \mathbb{P}^1)$ which gives us the number of inseparable (Δ, S)-semi-integral points on $\mathrm{Orb}^{d'}(C, \mathbb{P}^1)$ that we should neglect.

Therefore, if we would like to have any chance to be able to state a conjecture for function fields such as Conjecture 2.5 we must also take the Frobenius morphism, which gives us these Frobenius orbifold points, into account.

3.3 Conjecture for function fields

Concluding sections 3.1 and 3.2 we give a function field analogue of Conjecture 2.5 for function fields with characteristic $p > 0$ in this section.

The result from [12] from section 3.1 gave us immediately that for a non-isotrivial curve C of genus $g(C) \geq 2$ we will only expect finitely many rational points and thus we already know that we can only expect finitely many (Δ, S)-semi-integral points for any divisor Δ on (C, Δ). If C is isotrivial, the same holds if we do not count the constant sections and the Frobenius orbifold points. Therefore, the question about the number of (Δ, S)-semi-integral points on these Campana orbifolds of curves with genus greater than or equal to 2 is solved already: this number is finite.

This leaves the problem open to two cases:

- $X = \mathbb{P}^1$, which can have χ greater than, equal to, or less than zero depending on Δ,

- $X = C$ is an elliptic curve, which can have χ equal to zero only when Δ is empty, and in any other case $\chi < 0$.

Taking into account the issues we found with isotriviality and the Frobenius morphism, we state the conjecture over function fields as follows.

Conjecture 3.6. Let X be a curve over a function field F of transcendence degree 1 over a field k of characteristic $p \neq 0$. Let $\Delta = \sum \left(1 - \frac{1}{m_i}\right)[\sigma_i]$ be a divisor with m_i integers and $\sigma_i \in X$. After a possible finite extension k'/k and enlargement of the finite set S, the number of (Δ, S)-semi-integral points of bounded height d on the Campana orbifold (X, Δ), for any model, is asymptotic to

$q^{d \cdot \chi_\Delta}$	$\chi(X, \Delta) > 0$
$d^{\mathcal{O}(1)}$	$\chi(X, \Delta) = 0$
finite	$\chi(X, \Delta) < 0$

In the case where X is isotrivial to a curve X_k over k or to a curve $X_{\mathbb{F}}$ where $\mathbb{F}$ is a finite field, we do not count the constant sections or the Frobenius orbifold points.

In particular the number of (Δ, S)-semi-integral points of degree d on the squareful Campana curve $(\mathbb{P}^1, \Delta)$ with $\Delta = \frac{1}{2}[0] + \frac{1}{2}[1] + \frac{1}{2}[\infty]$ should be asymptotic to

$$q^{\frac{d}{2}}.$$

It is important to note that in the case where $\chi(X, \Delta) > 0$ (which is the main focus) we are always looking at an isotrivial Campana curve $(\mathbb{P}^1, \Delta)$ with Δ supported at at most three points. We expect $q^{d\chi_\Delta}$ (Δ, S)-semi-integral points of degree d on this Campana orbifold. We also expect a certain number of Frobenius orbifold points of degree d, which would come from maps of degree d/p. We will later show that there are $\mathcal{O}(q^{2d/p+1-g(C)})$ of such maps $f : C \to \mathbb{P}^1$ of degree d/p. Thus, if $d \cdot \chi > 2d/p$ or equivalently $p > 2/\chi$, the amount of Frobenius orbifold points is negligible in comparison to the amount of (Δ, S)-semi-integral points. Especially in the squareful case, where $\chi(\mathbb{P}^1, \Delta) = \frac{1}{2}$, this implies that when $p \geq 4$, we get a negligible amount of Frobenius orbifold points.

The remaining sections of the thesis will focus on this conjecture, mainly where $C = \mathbb{P}^1$ with three points in the support of Δ. First we will look at some small results for the different values of χ. Then we look at a data set which was calculated by Wieb Bosma for the 'squareful' case where $\chi = 1/2 > 0$. Finally we use an abstract approach using Hilbert schemes and Hom schemes in the general case when $\chi > 0$.

4 Results and Data on the Conjecture

In this section we try to get results on conjecture 3.6. We start with the case where $\chi(\mathbb{P}^1, \Delta) < 0$, where we expect a finite number of non-constant (Δ, S)-semi-integral points . We can prove a specific case of this result following Campana's article [4], however the full case is unsolved and much harder and therefore out of the scope of this thesis. We briefly discuss the case where $\chi_\Delta = 0$ and then turn our attention to the case where $\chi_\Delta > 0$, with special attention for the squareful Campana curve with $\Delta = \frac{1}{2}[0] + \frac{1}{2}[1] + \frac{1}{2}[\infty]$ and $\chi_\Delta = \frac{1}{2}$.

First, let us introduce a partial order on the divisors Δ. This will make working with Campana curves and their (Δ, S)-semi-integral points a bit easier.

Definition 4.1. Let $\Delta = \sum(1 - 1/m_i)\sigma_i$ and $\Delta' = \sum(1 - 1/m'_j)\tau_j$ be two divisors. We write $\Delta < \Delta'$ if $\Delta \neq \Delta'$ and for every σ_i in the support of Δ we have that $\sigma_i = \tau_j$ for some j with $m_i \leq m'_i$. We write $\Delta \leq \Delta'$ if we want to include the possibility that they are equal.

The following properties are then clear

Proposition 4.2. *If $\Delta' \leq \Delta$, then*

- $\deg \Delta' \leq \deg \Delta$,
- $\chi'_\Delta \geq \chi_\Delta$,
- *the set of (Δ, S)-semi-integral points on $(\mathbb{P}^1, \Delta)$ is a subset of the set of (Δ', S)-semi-integral points on $(\mathbb{P}^1, \Delta')$.*

4.1 The case $\chi(\mathbb{P}^1, \Delta) < 0$

As $g(\mathbb{P}^1) = 0$, it is impossible to get $\chi(\mathbb{P}^1, \Delta) = 2 - 2g - \deg \Delta < 0$ when $\deg \Delta \leq 2$. Therefore, the cases where $\chi(\mathbb{P}^1, \Delta) < 0$ will have at least three points in the support of Δ. A quick calculation using proposition 4.2 then gives us that there are three different cases to get $\chi(\mathbb{P}^1, \Delta) < 0$.

1. $\Delta \geq \Delta'$ where Δ' is supported on three points with
$$\left(\frac{1}{m_0} + \frac{1}{m_1} + \frac{1}{m_\infty}\right) < 1,$$
2. $\Delta \geq \Delta'$ where Δ' is supported on four points with $(m_i)_i = (2, 2, 2, 3)$,
3. $\Delta \geq \Delta'$ where Δ' is supported on five points with $(m_i)_i = (2, 2, 2, 2, 2)$.

We will give a proof of the finiteness of non-constant separable (Δ, S)-semi-integral points in case 1 below. For cases 2 and 3, Campana gives a solution over complex function fields in [4]. By Proposition 4.2 we can restrict ourselves to the cases where the boundary divisor is Δ'. However, this problem is much too hard to solve in this thesis, as both the use of Belyi maps (which worked over number fields in section 2.6) and the approach by Campana fail in positive characteristic.

Returning to case 1, we suppose that Δ is supported at [0], [1] and [∞]. Thus in this case $(\mathbb{P}^1, \Delta)$ is isotrivial and

$$\chi_\Delta = 2 - \left(1 - \frac{1}{m_0}\right) - \left(1 - \frac{1}{m_1}\right) - \left(1 - \frac{1}{m_\infty}\right) = \left(\frac{1}{m_0} + \frac{1}{m_1} + \frac{1}{m_\infty}\right) - 1 < 0$$

Campana uses the Riemann-Hurwitz theorem, however we will use the fact that the *abc*-theorem is proven for function fields. We use [16, Theorem 7.17]. Let us first describe the problem in the following way, which explains the link to the problem with squareful numbers (although note that the squareful Campana curve has $\chi(\mathbb{P}^1, \Delta) > 0$).

Let f be a (Δ, S)-semi-integral point on the squareful Campana curve, i.e. the curve $\mathbb{P}^1$ with $\Delta = \frac{1}{2}[0] + \frac{1}{2}[1] + \frac{1}{2}[\infty]$. Let f_0 denote the zero divisor of f, f_1 denote the zero divisor of $1 - f$ and f_∞ denote the zero divisor of $1/f$. (Or, think of these f_i as the inverse images of i, written as divisors). Then we can write

$$f_0 = \sum e_P P, \qquad f_0 = \sum e_Q Q, \qquad f_0 = \sum e_R R.$$

These e_P, e_Q, e_R are non-negative integers and because f is a (Δ, S)-semi-integral point on $(\mathbb{P}^1, \Delta)$, we must have $e_P, e_Q, e_R \neq 1$ for all P,Q and R (so that they are either 0 or at least 2). As $f + (1 - f) = 1$ we get that

$$\frac{f_0}{f_\infty} + \frac{f_1}{f_\infty} = 1$$

and thus

$$f_0 + f_1 = f_\infty$$

where each f_i is squareful. The degree of f is equal to the degree of these divisors.

In the general case for Δ with $\chi(\mathbb{P}^1, \Delta) < 0$, we can assume that $e_P \geq m_0$, $e_Q \geq m_1$ and $e_R \geq m_\infty$. With this notation, the *abc* theorem is applicable:

Theorem 4.3 (*abc* theorem)**.** *Suppose that there are degree d morphisms $f, g : C \to \mathbb{P}^1$ satisfying*

$$f + g = 1$$

and f, g separable. Then

$$d \leq 2g(C) - 2 + \sum_{P \in f_0, f_1, f_\infty} \deg P.$$

This theorem is applicable in this situation, because if $f + g = 1$ then $g = 1 - f$, this equation implies that

$$\frac{f_0}{f_\infty} + \frac{f_1}{f_\infty} = 1,$$

where f_0 denotes the zero divisor of f, f_1 denotes the zero divisor of $1-f$ and f_∞ denotes the zero divisor of $1/f$. Thus $f_0 + f_1 = f_\infty$, which implies that a (Δ, S)-semi-integral point f must adhere to the above inequality.

Theorem 4.4. *Let Δ be such that $\chi(\mathbb{P}^1, \Delta) < 0$ with Δ supported at 0, 1 and ∞. Then there are only finitely many separated (Δ, S)-semi-integral points on the Campana curve $(\mathbb{P}^1, \Delta)$.*

Proof. First, let $S = \emptyset$. Suitable (Δ, S)-semi-integral points f in this case would have all ramification indices of f_0 to be 0 or larger than m_0 and therefore we get

$$\sum_{P \in f_0} \deg P \leq \frac{d}{m_0}.$$

The same holds for f_1 and f_∞. This gives us that

$$d + 2 - 2g(C) \leq \sum_{P \in f_0, f_1, f_\infty} \deg P \leq d \left(\frac{1}{m_0} + \frac{1}{m_1} + \frac{1}{m_\infty} \right) \tag{3}$$

and thus

$$1 + \frac{2 - 2g(C)}{d} \leq \left(\frac{1}{m_0} + \frac{1}{m_1} + \frac{1}{m_\infty} \right) < 1.$$

As $2 - 2g(C)$ is the Euler characteristic $\chi(C)$, this happens only when

$$\frac{\chi(C)}{d} \leq \chi(\mathbb{P}^1, \Delta) < 0$$

and therefore

$$d \leq \frac{\chi(C)}{\chi(\mathbb{P}^1, \Delta)}.$$

As there are only finitely many separable morphisms of a certain degree, since the residue field is finite, we find indeed that there are only finitely many separable (Δ, S)-semi-integral points on a Campana curve $(\mathbb{P}^1, \Delta)$ with $\chi(\mathbb{P}^1, \Delta) < 0$ and Δ supported on three points, when $S = \emptyset$.

Now if $S = \{S_1, \ldots, S_k\}$, we can no longer assume that $\sum_{P \in f_i} \deg P \leq d/m_i$. However, if we write

$$f_i = \sum_{P \in f_i} e_p P = \sum_{P \notin S} e_P P + e_1 S_1 + \ldots e_k S_k,$$

we can assume that every $e_P \geq m_i$ and $e_i \geq 0$ for $1 \leq i \leq k$. Thus

$$d = \sum_{P \in f_i} e_p \deg P \geq \sum_{P \notin S} e_P \deg P$$

and thus $\sum_{P \in f_i \setminus S} \deg P \leq d/m_i$ where the index runs over the points in the support of f_i that are not in S. As S is a finite set, $\sum_S \deg S_i = k$ for some positive integer k. Again, we get

$$\begin{aligned} d + \chi(C) &\leq \sum_{P \in f_0, f_1, f_\infty} \deg P \\ &\leq \sum_{P \in S} + \sum_{P \in f_0 \setminus S} \deg P + \sum_{P \in f_1 \setminus S} \deg P + \sum_{P \in f_\infty \setminus S} \deg P \\ &\leq k + d \left(\frac{1}{m_0} + \frac{1}{m_1} + \frac{1}{m_\infty} \right) \end{aligned}$$

Thus with the same calculation as before we now get the inequality

$$d \leq \frac{\chi(C) - k}{\chi(\mathbb{P}^1, \Delta)}.$$

We conclude that the set of separated (Δ, S)-semi-integral points on the Campana curve $(\mathbb{P}^1, \Delta)$ is finite. □

4.2 The case $\chi(\mathbb{P}^1, \Delta) = 0$

When $\chi(\mathbb{P}^1, \Delta) = 0$, this implies that $\deg \Delta = 2$. When Δ is supported on three points, this comes down to the calculation

$$\left(1 - \frac{1}{m_0}\right) + \left(1 - \frac{1}{m_1}\right) + \left(1 - \frac{1}{m_infty}\right) = 2$$

and thus

$$\frac{1}{m_0} + \frac{1}{m_1} + \frac{1}{m_\infty} = 1.$$

When Δ is supported on four points, we only get the minimal solution where $(m_i)_i = (2, 2, 2, 2)$. If Δ is supported on five points will always have a larger degree than $\frac{5}{2}$. Thus we have the following possibilities for Δ

1. Δ is supported on three points with $(m_i)_i = (3, 3, 3)$,
2. Δ is supported on three points with $(m_i)_i = (2, 3, 6)$,
3. Δ is supported on three points with $(m_i)_i = (2, 4, 4)$,
4. Δ is supported on four points with $(m_i)_i = (2, 2, 2, 2)$.

In all of these cases, not much is known (although the first three are isotrivial). A naive analogue of conjecture 2.5 would suggest about $d^{\mathcal{O}(1)}$ points. This is supported by the fact that we can look at (4) as the image of the set of rational points under the map $E \to \mathbb{P}^1$, where E is an elliptic curve. The base divisor Δ_f on $\mathbb{P}^1$ can be written as

$$\frac{1}{2}[0] + \frac{1}{2}[1] + \frac{1}{2}[\lambda] + \frac{1}{2}[\infty]$$

for some $\lambda \in \mathbb{P}^1$. By Proposition 2.4, assuming such a map extends to the models, this would give us that every rational point on E gives us a (Δ, S)-semi-integral point on $(\mathbb{P}^1, \Delta)$. This provides us with some support for the conjecture, although it could be the case that there are a lot of (Δ, S)-semi-integral points on $(\mathbb{P}^1, \Delta)$ that do not lift to E. For example, every point that comes from a rational point of E intersects $[0]$, $[1]$, $[\lambda]$ and $[\infty]$ with even multiplicity, thus every point with an odd intersection multiplicity (unequal to 1) with any of these divisors does not lift to E. As of yet, we have no knowledge on the number of these points, although all of these points would lie on (different) Campana curves with $\chi(\mathbb{P}^1, \Delta) < 0$, so it seems unlikely that many such (Δ, S)-semi-integral points exist.

4.3 The case $\chi(\mathbb{P}^1, \Delta) > 0$

Let us now look at the case where $\chi_\Delta > 0$ and Δ is supported at 0, 1 and ∞ with multiplicities m_0, m_1 and m_∞ (note that Δ cannot be supported at 4 points, this would always give $\chi_\Delta \leq 0$). For the rest of this chapter, take $m_i = 2$, i.e. the squareful case. Note also, that we are now always working with isotrivial Campana curves.

Recall then that we want to count the functions which have a 'correct' intersection with the divisors $[0]$, $[1]$ and $[\infty]$. This means that the composition $f : C \to \mathcal{X} \to \mathbb{P}^1$ should have the 'correct' ramification above 0, 1 and ∞. Again, let f_0 denote the zero divisor of f, f_1 denote the zero divisor of $1 - f$ and f_∞ denote the zero divisor of $1/f$.

Our approach is as follows:

1. We calculate data for morphisms $\mathbb{P}^1 \xrightarrow{f} \mathbb{P}^1$ of degree d in the *ring* $\mathbb{F}_q[T]$, which will give us more insight into the problem over function fields. We do this in section 4.4.

2. We approach the problem from a more abstract angle using Hom-schemes. We look at the conjecture for all cases $C \xrightarrow{f} \mathbb{P}^1$. We do this in chapter 5.

First (small) conclusions

We can make some small conclusions for $\mathbb{P}^1 \to \mathbb{P}^1$ before we get started with the data. More specifically, we can make some constraints on the degrees d of possible squareful maps $\mathbb{P}^1 \to \mathbb{P}^1$.

A squareful $f : \mathbb{P}^1 \to \mathbb{P}^1$ should have ramification indices above 0,1, and ∞ either 0 or at least $m_0, m_1, m_\infty = 2$. If we write $f_0 = \sum e_P P$, we get that every e_P is larger than or equal to 2, and therefore $sum \deg P \leq d/2$ (and the same holds for f_1 and f_∞). Any P appears in at most one of these divisors, so we get that

$$\sum_{P \in f_0, f_1, f_\infty} \deg P \leq 3d/2.$$

Using the *abc* theorem 4.3, we get that

$$d \leq 2g_C - 2 + \sum_{P \in f_0, f_1, f_\infty} \deg P \leq 2g_c - 2 + 3d/2$$

so

$$4 - 4g_C \leq d.$$

We are looking at maps $\mathbb{P}^1 \to \mathbb{P}^1$, so $g(\mathbb{P}^1) = 0$ implies that $4 \leq d$ which explains why we won't get *any* separable $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree less than 4, with ramification indices at least 2.

In the specific case of $d = 5$ we must have $\sum_{P \in f_0} e_P \deg P = 5$. This can happen only when there are at most two points P_1 and P_2 in the support of f_0, because if there were three, one of these points would have to have ramification index 1. Therefore, the only possible ramification behavior for a squareful f is $f_i = 2 \cdot P_1 + 3 \cdot P_2$ with $\deg P_i = 1$. If we apply the *abc* theorem in this case, we get

$$d = 5 \leq -2 + \sum_{P \in f_0, f_1, f_\infty} \deg P = -2 + 6 = 4$$

which is not possible. Therefore we also won't find any separable $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree 5 with the correct ramification behavior.

We therefore know that there can be no squareful f of degree 2, 3 or 5.

4.4 Computational data

Let us now return to the approach using a data set. A small data set was calculated by Wieb Bosma for $f : \mathbb{P}^1 \to \mathbb{P}^1$ with base field $\mathbb{F}_5$ (the smallest field with characteristic $p \geq 5$) and $d \leq 10$, although coprimality of f_0, f_1 and f_∞ was not taken into consideration. We first list the main results, namely the degree d of f, the number of squareful f of degree d, denoted $n(d)$, the number of squareful f up to degree d, denoted $N(d) = \sum_{i \leq d} n(i)$ and the conjectured number $q^{d/2}$.

d	$n(d)$	$N(d)$	$q^d/2$
2	0	0	5
3	0	0	11
4	10	10	25
5	10	20	56
6	110	130	125
7	85	215	280
8	936	1.151	625
9	579	1.730	1.398
10	5.662	7.392	3.125

Although this is only a limited amount of data, it gives us some insight into the behavior of squareful f.

As expected, we find no f with degree 2,3 or 5, except for the 10 exceptional solutions in degree 5. We discuss these solutions in section 4.5. We have already briefly seen these in section 3.2.

Another interesting observation is the fact that $n(d)$ is "large" when d is even, and "much smaller" when $n(d)$ is odd. We cannot properly prove this behavior, although there is a heuristic argument, which fits the data.

4.4.1 A heuristic argument

Notice that the squareful divisors f_i can be written as $2 \cdot X_i + 3 \cdot Y_i$ with X_i an effective divisor and Y_i squarefree (i.e. for all P in the support of Y_i we have $e_P = 1$) in a unique way. For example, say

$$f_0 = 3P_1 + 5P_2 + 2P_3$$

we would write

$$X_0 = P_2 + P_3$$

$$Y_0 = P_1 + P_2.$$

For f_0 of degree d we can quite easily calculate the number of possible X_0 and Y_0, therefore we can calculate the total number of squareful divisors of a certain degree d. For the number of X_i we use the following lemma (see the discussion after Lemma 5.8 in [16]).

Lemma 4.5. *Let h_C denote the class field number. If $d > 2g_C - 2$, the number of effective divisors of degree d, denoted by b_d, equals*

$$h_C \frac{q^{d-g_C+1} - 1}{q - 1}$$

and thus $b_d = \mathcal{O}(q^d)$. In particular, in the case $C = \mathbb{P}^1$ we get that

$$b_d = \frac{q^{d+1} - 1}{q - 1}.$$

For the number of Y_i, we use proposition 2.3 from the same book:

Lemma 4.6. *Let c_d be the number of square-free monic polynomials in $\mathbb{F}_q[t]$ of degree d. Then $c_1 = q$ and $c_d = q^d(1 - \frac{1}{q})$ and thus $c_d = \mathcal{O}(q^d)$.*

Combining these facts, we see that we can easily estimate the number of squareful f_0 of degree d: first, write

$$d = 2\alpha + 3\beta$$

for some positive integers α and β. A squareful f_0 of degree d is than a combination of one of the b_α effective divisors and one of the c_β square-free divisors, leading to a total of

$$\frac{q^{\alpha+1}-1}{q-1} \cdot q^\beta \left(1 - \frac{1}{q}\right) = (q^{\alpha+1}-1) \cdot q^{\beta-1}$$

squareful divisors.

Thus, we see that most squareful divisors of degree d will have a decomposition of d with β as small as possible (either 0 or 1). Furthermore, if d is *even* we have a decomposition with $b = 0$, leading to a large number of squareful f_0. If d is *odd*, we will need at least $b = 1$, thus giving us relatively few squareful f_0. Although this heuristic approach fits the data so far, it is not too useful for the rest of the thesis. Therefore we leave it at this, and will not explore it any further. The decomposition of d into sums of 2 and 3 will however make a return in chapter 5, which will fit into this heuristic argument.

4.5 Exceptional solutions

In the data for $q = 5$ we find the expected 10 exceptional solutions in degree 5 and another 300 exceptional solutions in degree 10. We will display and explain these 300 solutions, which will provide more insight into what happens when the characteristicc of $\mathbb{F}_q$ divides the degree d. Let us use the following notation: X_i denotes a prime polynomial of degree 1 and Y_i denotes a prime polynomial of degree 2. Notice that there are 5 prime polynomials of degree 1 and 10 of degree 2. Then the exceptional solutions we get for $q = 5$ and $d = 10$ are listed in the following way:

- There are 10 of the form
$$X_1^{10} + X_2^{10}.$$
We can explain this because there are 5 options for X_1 and 4 for X_2 which, corrected for symmetry give $\frac{5 \cdot 4}{2} = 10$ solutions.

- There are 50 of the form
$$X_1^{10} + (X_2 X_3)^5.$$
We can explain this because there are 5 options for X_1, and 10 options for $X_2 X_3$ (as before) giving $5 \cdot 10 = 50$ options.

- There are 50 of the form
$$X_1^{10} + Y_1^5.$$
We can explain this because there are 5 options for X_1, and 10 options for Y_1 giving $5 \cdot 10 = 50$ options.

- There are 45 of the form
$$(X_1X_2)^5 + (X_3X_4)^5.$$
We can explain this because there are 10 options for X_1X_2, and then 10 options for X_3X_4 giving $10 \cdot 9/2 = 45$ options when corrected for symmetry (with coprime solutions).

- There are 45 of the form
$$Y_1^5 + Y_2^5.$$
We can explain this because there are 10 options for Y_1, and then 9 options for Y_2 giving $10 \cdot 9/2 = 45$ options when corrected for symmetry.

- There are 100 of the form
$$(X_1X_2)^5 + Y_1^5.$$
We can explain this because there are 10 options for X_1X_2, and 10 options for Y_1 giving $10 \cdot 10 = 100$ options when corrected for symmetry.

All of these give squareful results, because $(a+b)^5 = a^5 + b^5$ in characteristic 5. It is clear that from the above, we can take any effective divisor of degree 2 raised to the fifth power and add any other effective divisor of degree 2 raised to the fifth power. We can of course do this in a more general setting, where we now *do* account for coprimality of f_0, f_1 and f_∞.

Let d be a multiple of the characteristic of the finite field, p. Then we have that
$$f_0^p + f_1^p = (f_0 + f_1)^p$$
and therefore, *any* two coprime f_0, f_1 of degree $\delta := d/p$ will give us a exceptional squareful f. For this case (where we had $f_i \in \mathbb{F}_q[X]$), this implies that we can pick any of the q^δ possibilities for f_0 and f_1, and look at those cases where f_0 and f_1 are coprime. This has been done in [2, Theorem 3]. We will reproduce the relevant theorem 3 and proof here.

Theorem 4.7. *Let $\mathbb{F}_q$ be the finite field of q elements and let a and b be random polynomials of degree δ in $\mathbb{F}_q[x]$. Then the probability that a and b are coprime is $1 - 1/q$.*

Proof. The proof relies on a reversed Euclid's algorithm (jokingly named 'dilcuE's algorithm') for polynomials. Recall that in Euclid's algorithm we use that
$$a = q_1 b + r_1$$
(for polynomials q_1, r_1 with $\deg r_1 < \deg b$) and repeat this on
$$b = q_2 r_1 + r_2$$
until at last we get
$$r_{n-1} = q_{n+1} r_n + r_{n+1}$$

with $r_{n+1} \in \mathbb{F}_q$. If $r_{n+1} = 0$, then a and b have greatest common divisor r_n. If $r_{n+1} \neq 0$, they are coprime.

We can then apply the reversed algorithm (assuming we know all q_i), starting with r_n and r_{n+1} to reproduce all r_i and the original a and b. We will use this to show that for any pair (a, b) that are *not* coprime, we can construct $q-1$ pairs (a', b') that *are*.

So, if we start with any pair a and b which are *not* coprime, we get the unique set of q_i and r_i by the algorithm

$$(a, b) \xrightarrow{q_1} (b, r_1) \xrightarrow{q_2} \dots \xrightarrow{q_{n+1}} (r_n, 0)$$

with $r_{n+1} = 0$ as a and b are not coprime. Then for any $\alpha \in \mathbb{F}_q^*$ we can use the reversed algorithm,

$$(r_n, \alpha) \xrightarrow{q_{n+1}} (r'_{n-1}, r_n) \xrightarrow{q_n} \dots \xrightarrow{q_1} (a', b')$$

to obtain a unique coprime pair a' and b'. So any pair (a, b) that is not coprime gives us $q-1$ unique pairs (a', b') that are coprime. We conclude that the probability for any pair (a, b) of degree δ to be coprime equals $\frac{q-1}{q} = 1 - 1/q$. $\square$

By the theorem, we can conclude that when the degree d is a multiple of the characteristic, i.e. $d = \delta p$, we get q^δ options for f_0 and f_1 with a chance of $1 - 1/q$ to be coprime. This gives us (as validated by the data)

$$q^{2\delta}(1 - \frac{1}{q})$$

exceptional Frobenius orbifold points f on the Campana curve. Note that because $\delta = d/p$ this gives us

$$\mathcal{O}\left(q^{2d/p}\right)$$

Frobenius orbifold points on the Campana curve of degree d which we do not want to count. As the conjecture gives

$$\mathcal{O}\left(q^{\frac{d}{2}}\right),$$

when $p \geq 5$, we find that these exceptional solutions do not add enough points to alter the conjecture. In the following section, we will see that this argument can be generalised to find the number of Frobenius orbifold points of the form $f : C \to \mathbb{P}^1$.

4.5.1 Link to ζ-function

It has long been known that the probability of two positive integers below a certain large integer N being coprime tends to

$$\frac{1}{\zeta(2)}$$

as $N \to \infty$, where

$$\zeta(s) = \sum_n \frac{1}{n^s}$$

is the Riemann zeta function (see for example the explanation in [19]). In fact, there is the more general result from [17, p. 447] that in any number field K and for any $n > 1$ the probability that n ideals with norms below a certain large integer N are relatively prime tends to

$$\frac{1}{\zeta_K(n)}$$

as $N \to \infty$ where ζ_K is the Dedekind zeta function of K,

$$\zeta_K(s) = \sum_{I \subset \mathcal{O}_K} \frac{1}{\left(N_{\mathbb{Q}}^K(I)\right)^s}.$$

Here $N_{\mathbb{Q}}^K$ denotes the norm of K and I is a non-zero ideal of $\mathcal{O}_K$. This $\zeta_K(s)$ converges when the real part of s is greater than 1. In that case, we can rewrite it as an Euler product to get

$$\zeta_K(s) = \prod_{\mathfrak{p} \text{ prime}} \frac{1}{1 - N_{\mathbb{Q}}^K(\mathfrak{p})^{-s}}.$$

It should therefore come as no surprise that the factor $1 - 1/q$ we found in theorem 4.7 is precisely

$$\frac{1}{\zeta_A(2)}$$

where $A = \mathbb{F}_q[T]$ and ζ_A is the zeta function

$$\zeta_A(s) = \sum_{\text{monic } f \in A} \frac{1}{|f|^s}$$

Here $|f| = q^{\deg f}$ denotes the norm of f. This sum converges, when the real part of s is greater than 1, to

$$\frac{1}{1 - q^{1-s}}.$$

In [18, p.19] it is explained why this function field analogue also holds for elements in the function field $K(C)$ with regards to the zeta function $\zeta_{K(C)}$. From [16] we get that the number of effective divisors of degree d over a function field $K(C)$ of genus g and class number h_K is equal to

$$b_d := h_K \frac{q^{d-g+1} - 1}{q - 1.}$$

This gives us that we should expect about

$$h_K \frac{q^{2(d-g+1)}}{(q-1)\zeta_K(2)}$$

Frobenius orbifold points on the Campana curve of degree dp. By Theorem 5.9 and Proposition 5.11 from [16, Lemma 5.8] we find that $h_K = \mathcal{O}(q^g)$ and $1/\zeta_K(2) = \mathcal{O}(1)$ which implies that we would expect about $\mathcal{O}(q^{2d+1-g(C)})$ Frobenius orbifold points of degree dp. The conjecture for the number of (Δ, S)-semi-integral points on the squareful Campana curve $(\mathbb{P}^1, \Delta)$ with $\Delta = \frac{1}{2}[0] + \frac{1}{2}[1] + \frac{1}{2}[\infty]$ and $\chi(\mathbb{P}^1, \Delta) = \frac{1}{3}$ of degree dp predicts

$$\mathcal{O}(q^{dp/2}).$$

As $p/2 > 2$ for $p \geq 5$ this implies that there will be a negligible amount of Frobenius orbifold point in the total amount of (Δ, S)-semi-integral points (if the conjecture holds), when $p \geq 5$. This agrees with the condition that p should be larger than $2/\chi$ that we found in the end of section 3.6. Only for fields of characteristic 2 or 3, do we expect too many Frobenius orbifold points in our data, in comparison with non-Frobenius (Δ, S)-semi-integral points.

5 Abstract approach

In this section we use an abstract approach to study conjecture 3.6 and hope to explain some of the things we observed in the data in the previous section. Our main objects of interest are the Campana curves $(\mathbb{P}^1, \Delta)$ with $\chi(\mathbb{P}^1, \Delta) > 0$, especially the 'squareful' curve with $\Delta = \frac{1}{2}[0] + \frac{1}{2}[1] + \frac{1}{2}[\infty]$. We will use Hom-schemes (and Hilbert schemes) for this approach. We will also look at an interesting map into the symmetric product of a curve. For this, some basic knowledge from [8] is expected. We will not be able to fully explore this approach, but hope that it sketches an approach for further research.

5.1 Hilbert schemes and Hom-schemes

To start, we will need to describe the Hom-scheme, which is a scheme that describes the morphisms between projective schemes $X \to Y$. This description is derived from the thesis of Drost [5] and chapter 1 and 2 of [10].

In these works one can read in detail how the *Hom*-scheme of morphisms $X \to Y$, denoted $\mathrm{Hom}(X, Y)$ can be derived as a subscheme of the Hilbert scheme $\mathrm{Hilb}(X \times_S Y)$, a scheme in which the underlying space contains all closed subschemes of $X \times Y$. We only need a small part of this theory, namely:

Definition 5.1. The *Hilbert scheme* of a projective scheme X over S is defined to be the scheme that represents the functor

$$Hilb(X) : \{\text{schemes over } S\} \to \{\text{sets}\}$$

$$Z \mapsto \{\text{subschemes of } X \times_S Z \text{ proper and flat over } Z\}.$$

This scheme $\mathrm{Hilb}(X)$ has the property that it is projective over S. Furthermore we define:

Definition 5.2. The Hom-*scheme* between two varieties X and Y over S is defined to be the scheme that represents the functor

$$Hom(X, Y) : \{\text{schemes over } S\} \to \{\text{sets}\}$$

$$Z \mapsto \mathrm{Hom}_Z(X \times_S Z, Y \times_S Z).$$

The Hom-scheme is an open subscheme of $\mathrm{Hilb}(X \times_s Y)$, indeed one obtains an embedding by associating to such a morphism its graph.

Example 5.3. If X and Y are both the scheme $\mathbb{P}^1$ over some field k, then $\mathrm{Hom}(\mathbb{P}^1, \mathbb{P}^1)$ is the disjoint union of all subschemes $\mathrm{Hom}^d(\mathbb{P}^1, \mathbb{P}^1)$ of morphisms $\mathbb{P}^1 \to \mathbb{P}^1$ of degree d. These morphisms can be written down: a degree d morphism $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ maps $(x : y)$ to (ϕ_1, ϕ_2) with ϕ_1, ϕ_2 homogeneous polynomials of degree d in variables x and y. Equivalently, we can see the affine part of ϕ as a fraction of two coprime polynomials in x, of which the largest degree is d. Such a morphism is determined by the d zeroes of ϕ_1, the d zeroes of ϕ_2 and the correct scalar. If we count these zeroes with multiplicity we get $2d+1$ points that determine ϕ. Therefore, the set of these morphisms of degree d is isomorphic to an affine subscheme of $\mathbb{P}^{2d+1}$.

Another subscheme we need is $\mathrm{Hom}(X, Y; g)$ for $g : Z \to Y$ where $Z \subset X$ is a closed subscheme, projective over S. This subscheme $\mathrm{Hom}(X, Y; g)$ consists of all morphims $f : X \to Y$ such that $f|_Z = g$. If we let $Z = \{P_1, \ldots, P_k\}$ be the reduced scheme supported on k points of X, then we can prescribe where we want to map these points via g, say $g(P_i) = Q_i$. The subscheme $\mathrm{Hom}(X, Y; g)$ then consists of exactly those morphisms $f : X \to Y$ with $f(P_i) = Q_i$.

These schemes are interesting to us, because per definition of the Campana curve $(\mathbb{P}^1, \Delta)$ we were interested in certain maps between curves $C \to \mathbb{P}^1$ of degree d, i.e. elements of $\mathrm{Hom}^d(C, \mathbb{P}^1)$. Using the subscheme $\mathrm{Hom}(C, \mathbb{P}^1; g)$ could then give us some additional information on the behavior of these points as well. We denote the space of (Δ, S)-semi-integral points $f \in \mathrm{Hom}(C, \mathbb{P}^1)$ with $\mathrm{Orb}(C, \mathbb{P}^1)$, as we did in section 3. Again, if we want to specify the degree d, we write $\mathrm{Orb}^d(C, \mathbb{P}^1)$. Of course we have

$$\mathrm{Hom}(C, \mathbb{P}^1) = \bigsqcup_d \mathrm{Hom}^d(C, \mathbb{P}^1)$$

and we also have

$$\mathrm{Orb}(C, \mathbb{P}^1) = \bigsqcup_d \mathrm{Orb}^d(C, \mathbb{P}^1).$$

5.2 Local properties of the Hom-scheme

We are mostly interested in the local structure of the Hom-scheme. This structure is explained in chapter 2 of Kollár's book [10]. Although some properties hold for far more general situations, we will always let $X = C$ be a smooth curve over a finite field, and $Y = \mathbb{P}^1$. The following theorem, [5, Theorem 1.22], will be useful.

Theorem 5.4. *Let f be a morphism $C \to \mathbb{P}^1$. Let $\mathcal{T}_{\mathbb{P}^1}$ denote the tangent space of $\mathbb{P}^1$. The tangent space of $\mathrm{Hom}(C, \mathbb{P}^1)$ at $[f]$ is naturally isomorphic to*

$$\mathrm{Hom}_C(f^*\Omega_{\mathbb{P}^1}, \mathcal{O}_C).$$

Furthermore, the dimension of each irreducible component of $\mathrm{Hom}(C, \mathbb{P}^1)$ at a point $[f]$ is at least

$$\dim H^0(C, f^*\mathcal{T}_{\mathbb{P}^1}) - \dim H^1(C, f^*\mathcal{T}_{\mathbb{P}^1}). \tag{4}$$

A very similar theorem (II. Theorem 1.7, [10]) also tells us how this dimension changes when we prescribe the behavior of f on some closed subscheme $B \subset C$.

Theorem 5.5. *Let $B \subset C$ be a closed subscheme and let $g : B \to \mathbb{P}^1$ be a morphism. Let $\mathcal{I}_B$ denote the ideal sheaf of B. If $f : C \to \mathbb{P}^1$ satisfies $f|_B = g$, the dimension of the irreducible component of $\mathrm{Hom}(C, \mathbb{P}^1)$ at the point $[f]$ is at least*

$$\dim H^0(C, f^*\mathcal{T}_{\mathbb{P}^1} \otimes \mathcal{I}_B) - \dim H^1(C, f^*\mathcal{T}_{\mathbb{P}^1} \otimes \mathcal{I}_B). \tag{5}$$

Using Riemann-Roch we can rewrite (4), because in this setting we get

$$\begin{aligned}\dim H^0(C, f^*\mathcal{T}_{\mathbb{P}^1}) - \dim H^1(C, f^*\mathcal{T}_{\mathbb{P}^1}) &= \deg_C f^*\mathcal{T}_{\mathbb{P}^1} + \chi(\mathcal{O}_C) \\ &= -\deg_C f^*K_{\mathbb{P}^1} + \chi(\mathcal{O}_C),\end{aligned}$$

where $f^*K_{\mathbb{P}^1}$ is the divisor associated to the canonical bundle. We can also rewrite (5), using

$$\begin{aligned}\dim H^0(C, f^*\mathcal{T}_{\mathbb{P}^1} \otimes \mathcal{I}_B) &- \dim H^1(C, f^*\mathcal{T}_{\mathbb{P}^1} \otimes \mathcal{I}_B) \\ &= \deg_C(f^*\mathcal{T}_{\mathbb{P}^1} \otimes \mathcal{I}_B) + \chi(\mathcal{O}_C) \\ &= -\deg_C f^*K_{\mathbb{P}^1} - \ell(B) + \chi(\mathcal{O}_C)\end{aligned}$$

Furthermore, equality holds in both cases if $\deg f \geq 1 + g_C$, although this is harder to show (see [10, II, Thm. 1.7]).

As we know that $K_{\mathbb{P}^1} = -2 \cdot [\infty]$ (or any other point) and $\chi(\mathcal{O}_C) = 1 - g_C$ this all wraps up into the following theorem.

Theorem 5.6. *Let f be a morphism $C \to \mathbb{P}^1$ of degree d. Then*

$$\dim_{[f]} \operatorname{Hom}(C, \mathbb{P}^1) \geq 2d + 1 - g_C.$$

If $f|_B = g$ for some $g : B \to \mathbb{P}^1$, we get

$$\dim_{[f]} \operatorname{Hom}(C, \mathbb{P}^1; g) \geq 2d + 1 - g_C - \ell(B).$$

Equality holds when $d \geq 1 + g_C$.

For example, if we take $C = \mathbb{P}^1$, then $g(\mathbb{P}^1) = 0$ and thus d is always larger than or equal to $1 - g(\mathbb{P}^1)$. Therefore, for every $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree d, the local dimension of $\operatorname{Hom}(\mathbb{P}^1, \mathbb{P}^1)$ at f is equal to $2d + 1$. If we pick a number of points $P_1, \ldots, P_k$ and let $B = \{P_1, \ldots, P_k\}$ then $\ell(B) = k$. We can map these by some $g : B \to \mathbb{P}^1$ and look at the local dimension of $\operatorname{Hom}(\mathbb{P}^1, \mathbb{P}^1; g)$ which then equals $2d + 1 - k$.

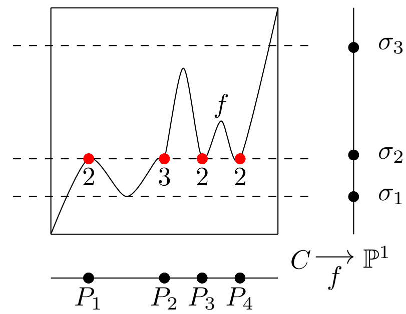
5.3 Symmetric product of curves

Consider the Hilbert scheme of d points on C, often denoted C^d, divided out by the action of S_d. We call this variety the symmetric product of d points on C, or $\operatorname{Sym}^d C$. Points on $\operatorname{Sym}^d C$ correspond to effective divisors of degree d on C, i.e. formal sums of points on C with positive integer coefficients which we write as $a_1P_1 + \ldots + a_nP_n$. We mostly follow [13, section 1.1 & 9.3].

Recall that we write $\Delta = \sum(1 - \frac{1}{m_i})\sigma_i$. Now for any $f \in \operatorname{Hom}^d(C, \mathbb{P}^1)$ we can look at the intersection with one of the divisors σ_i of the Campana curve $(\mathbb{P}^1, \Delta)$. We get a map into $\operatorname{Sym}^d(C)$ given by

$$\begin{aligned}\phi_i : \operatorname{Hom}^d(C, \mathbb{P}^1) &\to \operatorname{Sym}^d(C) \\ f &\mapsto f^*(\sigma_i \cdot f(C)).\end{aligned}$$

For example, the f of degree $d = 9$ in the picture below would map to $2P_1 + 3P_2 + 2P_3 + 2P_4 \in \operatorname{Sym}^9 C$ under ϕ_2.



In $\mathrm{Sym}^d(C)$ we want to look at the subspace Eq_ω where ω is a partition of d, i.e. a set $\omega = (m_1, \ldots, m_l)$ with $m_1 + \ldots m_l = d$ and $m_i \leq m_j$ if $i \leq j$. We define a *stratum* Eq_ω in $\mathrm{Sym}^d(C)$ by

$$\mathrm{Eq}_\omega := \Big\{ m_1 P_1 + \ldots m_l P_l \in \mathrm{Sym}^d C \mid P_i \neq P_j \text{ when } i \neq j \Big\}.$$

Example 5.7 (The squareful case)**.** Our initial problem was counting the number of $f \in \mathrm{Orb}^d_\Delta(C, \mathbb{P}^1)$ on the Campana curve $(\mathbb{P}^1, \Delta)$ with $\Delta = \frac{1}{2}[0] + \frac{1}{2}[1] + \frac{1}{2}[\infty]$. Let us take some f of degree 7 in $\mathrm{Orb}^7(C, \mathbb{P}^1)$. Then this f would have to be mapped by $\phi_{[0]}$ into Eq_ω, where ω is one of $(2, 2, 3)$, $(2, 5)$, $(3, 4)$ or (7) (and of course the same holds for $\phi_{[1]}$ and $\phi_{[\infty]}$). Thus, the set of (Δ, S)-semi-integral points of degree 7, $\mathrm{Orb}^7(C, \mathbb{P}^1)$, is equal to the intersection of all pre-images under the maps ϕ_{σ_i} of these subspaces Eq_ω, where ω is a partition of d with all terms at least 2.

Definition 5.8. We define a partial ordening $\preceq$ on these partitions ω of d. We write $\omega' \preceq \omega$ if ω' can be obtained from ω by adding some m_i together. In other words, let $\omega = (m_1, \ldots, m_l)$ and $\omega' = (n_1, \ldots, n_k)$, then $\omega' \preceq \omega$ if and only if for all i

$$n_i = \sum_{j \in J_i} m_j$$

and $\bigcup_i J_i = \{1, \ldots, l\}$ with $J_i \cap J_j = \emptyset$ if $j \neq i$. So for example,

$$(7) \preceq (2, 5) \preceq (2, 2, 3).$$

Note that $\preceq$ is reflexive, so $\omega \preceq \omega$.

Using this partial ordening, the closure (in the sense of [13, section 1.1 & 9.3]) of Eq_ω in $\mathrm{Sym}^d(C)$ can be written as

$$\overline{\mathrm{Eq}_\omega} = \bigsqcup_{\omega' \preceq \omega} \mathrm{Eq}_{\omega'}.$$

We prove this in the following way. Assume $\omega' = (n_1, \ldots, n_k)$ is obtained from $\omega = (m_1, \ldots, m_k, m_{k+1})$ by $n_i = m_i$ for $i < k$ and $n_k = m_k + m_{k+1}$ (by induction, this will prove the result for all $\omega' \preceq \omega$). Let $P = \sum n_i P_i \in \mathrm{Eq}_{\omega'}$.

We can find two sequences of points on C, say (Q_n) and (R_n) that collide (in the sense of [13, section 1.1 & 9.3]) to P_k and thus the sequence of elements of Eq_ω

$$P_n = \sum_{i<k} n_i P_i + m_k Q_n + m_{k+1} R_n$$

collides to P in $\mathrm{Sym}^d(C)$. Thus any element of $\mathrm{Eq}_{\omega'}$ will be in the closure of Eq_ω if $\omega' \preceq \omega$. Disjointness follows per definition of Eq_ω. This also implies that if $\omega' \preceq \omega$, then $\overline{\mathrm{Eq}_{\omega'}} \subset \overline{\mathrm{Eq}_\omega}$.

Lemma 5.9. *The set* Eq_ω *is locally closed in* $\mathrm{Sym}^d(C)$, *i.e. it is the intersection of an open subset and a closed subset.*

Proof. It is enough to prove that Eq_ω is open in $\overline{\mathrm{Eq}_\omega}$. But this is easy, as

$$\overline{\mathrm{Eq}_\omega} \setminus \mathrm{Eq}_\omega = \left(\bigsqcup_{\omega' \preceq \omega} \mathrm{Eq}_{\omega'} \right) \setminus \mathrm{Eq}_\omega = \bigsqcup_{\omega' \prec \omega} \mathrm{Eq}_{\omega'} = \bigcup_{\omega' \prec \omega} \overline{\mathrm{Eq}_{\omega'}}$$

is a finite union of closed subsets of $\overline{\mathrm{Eq}_\omega}$. □

As every stratum Eq_ω is locally closed, this means that $\overline{\mathrm{Eq}_\omega}$ is a finite union of locally closed sets, and therefore *constructible*. We are interested in the following: we know that $\phi_i(\mathrm{Orb}^d(C, \mathbb{P}^1))$ is a subset of $\bigsqcup_\omega \mathrm{Eq}_\omega$ where $\omega = (m_1, \ldots, m_l)$ is a partition of d with $m_i \geq 2$ for all i. Looking at the closures of these, we clearly do not need all closed strata $\overline{\mathrm{Eq}_{\omega'}}$ as most of these are contained in another $\overline{\mathrm{Eq}_\omega}$ with $\omega' \preceq \omega$. In fact, we only need the finest partitions, i.e. a partition of d using only $m_i = 2$ or $m_i = 3$ for all i. In section 4.4.1 we already noted that the number of 2's and 3's needed in a partition of d determined the number of squareful divisors, where we found that there were more squareful divisors if the number of 3's was as small as possible. We therefore expect to see the dimension of $\overline{\mathrm{Eq}_\omega}$ for any such finest partitions ω grow if we use more 2's and (thus) lose 3's, as we saw in the data.

A partition $\omega = (m_1, \ldots, m_l)$ of d can also be written as $(\alpha_1^{e_1} \cdots \alpha_n^{e_n})$ where α_i is a natural number and e_i is the number of times α_i appears in the partition. (So $(2, 2, 2, 3)$ will be written $(2^3 \cdot 3^1)$ with $\alpha_1 = 2, e_1 = 3$ and $\alpha_2 = 3, e_2 = 1$). For such a partition, look at the morphism of algebraic varieties

$$\mathrm{Sym}^{e_1}(C) \times \ldots \times \mathrm{Sym}^{e_n}(C) \xrightarrow{\epsilon_\omega} \mathrm{Sym}^d(C)$$

$$(\rho_1, \ldots, \rho_n) \mapsto \alpha_1 \rho_1 + \ldots \alpha_n \rho_n$$

which is an isomorphism (see [11, p. 335]) onto the closed stratum

$$\mathrm{Sym}^{e_1}(C) \times \ldots \times \mathrm{Sym}^{e_n}(C) \xrightarrow{\sim} \overline{\mathrm{Eq}_\omega}.$$

In the case where ω is a partition of d with all terms either 2 or 3, we can write $\omega = (2^n, 3^m)$ for some n, m and as ω is a partition of the degree d we see that $2n + 3m = d$. By abuse of notation, we write ϵ_m instead of ϵ_ω when $\omega = (2^n, 3^m)$. For such an ω the map

$$\mathrm{Sym}^n(C) \times \mathrm{Sym}^m(C) \xrightarrow{\epsilon_m} \mathrm{Sym}^d(C)$$

has image $\mathrm{Im}(\epsilon_m) = \overline{\mathrm{Eq}_m}$ with dimension $n + m = \frac{d-m}{2}$.

5.4 The map ϕ

Recall that we were looking at the map

$$\phi_d : \mathrm{Hom}^d(C, \mathbb{P}^1) \to \mathrm{Sym}^d(C)$$

$$f \mapsto [\sigma] \cdot \Gamma_f$$

for $\sigma \in \mathbb{P}^1$. We will often write ϕ instead of ϕ_d when d is implicitly understood. Knowing a lot about ϕ, specifically in relation to Eq_ω, would give us a lot of knowledge about the ramification behavior of the maps $f : C \to \mathbb{P}^1$ which would lead to a greater understanding of the behavior of semi-integral points. We hope that this section and previous sections show that ϕ is an interesting function and therefore hope that this approach will be able to answer some basic questions about ϕ, although we will not be able to do so in this thesis.

Let us end with some ideas on the question: when is ϕ_d surjective?

We can certainly say that this is not always the case. Take for example C hyperelliptic. We know (for example from [8, IV.2 ex 2.2]) that the morphism $f : C \to \mathbb{P}^1$ of degree 2 is unique up to an automorphism of $\mathbb{P}^1$. The automorphism group of $\mathbb{P}^1$ is equal to $\mathrm{PGL}_q(2)$ so in this case ($d = 2$) the map $\phi_2 : \mathrm{Hom}^2(C, \mathbb{P}^1) \to \mathrm{Sym}^2(C)$ cannot be surjective.

We will formulate this question differently, assuming $\sigma = 0$: for which effective divisors D can we find a map $f : C \to \mathbb{P}^1$ such that the fiber above 0 is equal to D?

For the analogous question over $\mathbb{C}$, this is easy: any base point free divisor D (so certainly any D satisfying $\deg D > 2g$, see [8, IV, 3.2]) gives us that

$$\ell(D) = \ell(D - P) + 1$$

for all points P in D (using Riemann-Roch). A finite number of lower dimensional subspaces over $\mathbb{C}$ cannot cover the vector space $L(D)$ and thus $L(D) \setminus \bigcup_{P \in D} L(D - P)$ is non-empty. Any f in that set will have the correct fiber over 0.

This argument however, does not work in our case. A finite number of lower dimensional subspaces *can* cover a vector space over a finite field. Therefore, more theory is needed.

Let us look at $\mathrm{Hom}^d(C, \mathbb{P}^1)$ as an open subscheme of $\mathrm{Hilb}_{C \times \mathbb{P}^1}$, where $f : C \to \mathbb{P}^1$ corresponds to its graph Γ_f. View $\mathrm{Sym}^d(C)$ as the effective divisors in Div_C. We write Pic for the Picard group of divisors up to linear equivalence. We write J_C for the Jacobian of the curve C, the group of divisors of degree 0 up to linear equivalence.

By Theorem 3.3.12 of [21] we get the following.

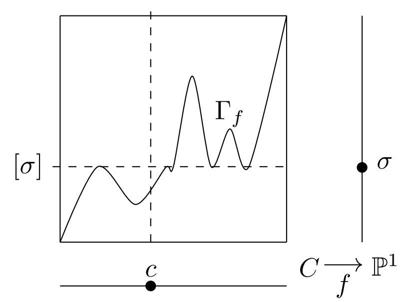
Theorem 5.10. *Let C and C' be curves. Then the following sequence is exact and splits*

$$0 \to \mathrm{Pic}_C \times \mathrm{Pic}_{C'} \to \mathrm{Pic}_{C \times C'} \to \mathrm{Hom}(J_C, J_{C'}) \to 0$$

As the Jacobian of $\mathbb{P}^1$ is trivial, in this case we get

$$\mathrm{Pic}_C \times \mathrm{Pic}_{\mathbb{P}^1} \xrightarrow{\sim} \mathrm{Pic}_{C\times\mathbb{P}^1}\,.$$

We can then identify Γ_f with some section s of a line bundle $\mathscr{M}$. We know that any line bundle on $\mathrm{Pic}_{\mathbb{P}^1}$ is of the form $\mathcal{O}(n)$ for some n and so $\mathscr{M} = \mathscr{L} \boxtimes \mathcal{O}(n)$. In this case, n must be 1 as the Γ_f must have only one vertical component (there is only one intersection above c for any $c \in C$). Using the same argument in the horizontal direction (looking at the intersection with $[\sigma]$), we find that $\mathscr{L} = f^*\mathcal{O}(1)$. So, we can identify f via Γ_f with some section s of $\mathscr{L} \boxtimes \mathcal{O}(1)$.



5.5 Discussion

This correspondence between morphisms f and pairs $(s, \mathscr{L})$ could yield valuable information about our map ϕ. It is useful to get a good understanding of the map ϕ which might enable one to compute the dimension of the orbifold locus of the Hom-scheme, or at least provide upper or lower bounds. This would yield valuable information on the distribution of the (Δ, S)-semi-integral points of bounded height, and more specifically on the distribution of semi-integral points on the squareful Campana orbifold.

For the case where $\chi = 0$ and $\chi < 0$, it is very hard to say anything at all. Most methods over number fields do not work anymore over function fields of positive characteristic (such as Belyi maps). In characteristic zero this is solved by Campana by using differential forms, however this again becomes problematic in positive characteristic. Possibly, a larger data set could give insight into the behavior of these morphisms, however, this is computationally difficult to do.

We therefore leave it at this. Having given a general overview of the known results on the number of semi-integral points on Campana orbifolds over number fields, and having stated the difficulties that arise for Campana orbifolds over function fields, we have stated our conjecture for the behavior of semi-integral points on Campana orbifolds over function fields. We have sketched an approach using Hom-schemes and the symmetric product of a curve that might be fruitful, but this turned out to be harder than expected.

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