Radboud University

The Matrix Code Equivalence problem

Hardness estimates in the rank metric

Krijn Reijnders (joint work with **Simona Samardjiska** and **Monika Trimoska**)
Radboud University

Workshop in Coding and Cryptography, March '22

Today's topic

- **Code Equivalence problem:**
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- In this talk:
  - **How hard is MCE?**
  - **How to solve it?**

Matrix Code Equivalence (MCE)

The Matrix Code Equivalence Problem

Matrix code $\mathcal{C}$: a subspace of $\mathcal{M}_{m\times n}(\mathbb{F}_q)$ of dimension k endowed with **rank metric**

$$d(\mathrm{A},\mathrm{B}) = \mathrm{Rank}(\mathrm{A}-\mathrm{B})$$

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Matrix Code Equivalence (MCE) problem [Berger, 2003]

MCE$(k,n,m,\mathcal{C},\mathcal{D})$:
Input: Two k-dimensional matrix codes $\mathcal{C},\mathcal{D}\subset\mathcal{M}_{m\times n}(\mathbb{F}_q)$
Question: Find – if any – an isometry $\mu:\mathcal{C}\to\mathcal{D}$.

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Question: Find – if any – an isometry $\mu:\mathcal{C}\to\mathcal{D}$.

Known: Any isometry $\mu:\mathcal{C}\to\mathcal{D}$ can be written, for some $\mathrm{A}\in\mathrm{GL}_m(q),\mathrm{B}\in\mathrm{GL}_n(q)$, as

$$\mathrm{C}\mapsto \mathrm{ACB}\in\mathcal{D}$$

Known results [Couvreur, Debris–Alazard & Gaborit, 2021]

$$\mu : \mathrm{C} \mapsto \mathrm{ACB} \in \mathcal{D}, \quad \text{with } \mathrm{A} \in \mathrm{GL}_m(q) \text{ and } \mathrm{B} \in \mathrm{GL}_n(q)$$

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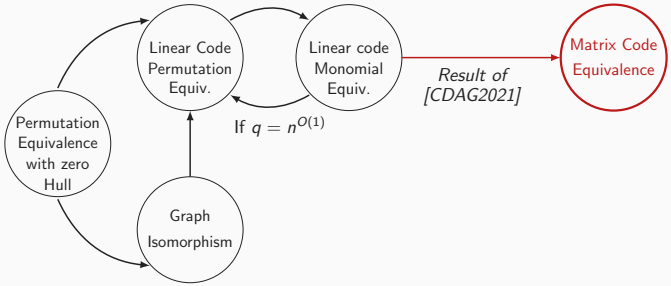
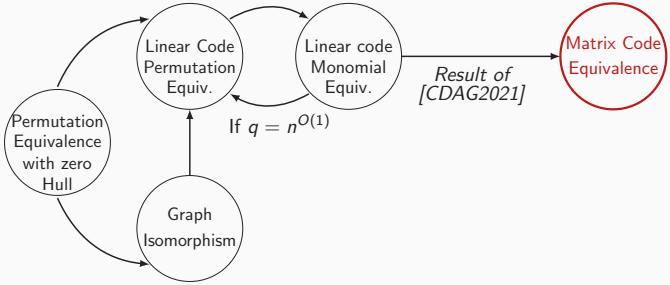


New results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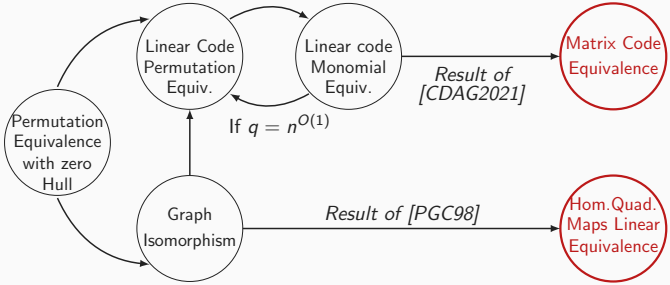

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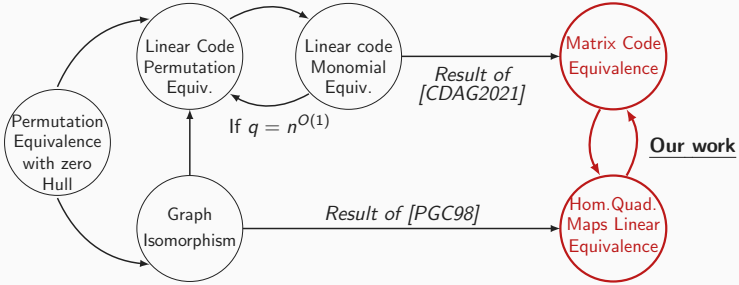

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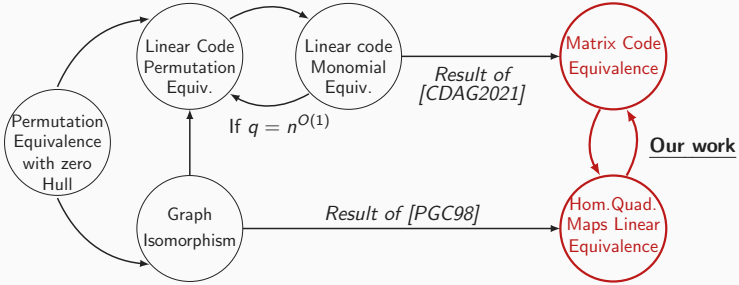

- Homogenous Quadratic Maps Linear Equivalence (hQMLE) problem is well known equivalence problem from multivariate crypto (instance of Isomorphism of Polynomials)
- We show that MCE **is equivalent to** hQMLE
- We upper bound nontrivially the complexity of solving MCE
  - solvable in $\mathcal{O}^*(q^{2/3(m+n)})$ time, for some cases can be improved to $\mathcal{O}^*(q^{\min(m,n)})$
  - previous upper bound $\mathcal{O}^*(q^{\min(m,n)^2})$ time: brute force one side, then solve MCRE

What is hQMLE?

Multivariate crypto basics

- systems of multivariate polynomials $\mathcal{P} = (p_1, p_2, \ldots, p_k)$, every p_s polynomial in N variables $x_1, \ldots, x_N$

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$$p_s(x_1, \ldots, x_N) = \sum \gamma_{ij}^{(s)} x_i x_j + \sum \beta_i^{(s)} x_i + \alpha^{(s)}, \qquad \alpha^{(s)}, \beta_i^{(s)}, \gamma_{ij}^{(s)} \in \mathbb{F}_q$$

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homogeneous Quadratic Maps Linear Equivalence (hQMLE) problem

hQMLE$(N, k, \mathcal{F}, \mathcal{P})$:

Input: Two k-tuples of multivariate polynomials

$\mathcal{F} = (f_1, f_2, \ldots, f_k)$, $\mathcal{P} = (p_1, p_2, \ldots, p_k) \in \mathbb{F}_q[x_1, \ldots, x_N]^k$

Question: Find – if any – $\mathsf{S} \in \mathrm{GL}_N(q), \mathsf{T} \in \mathrm{GL}_k(q)$ such that

$$\mathcal{P}(\mathrm{x}) = \mathcal{F}(\mathrm{xS}) \cdot \mathsf{T}$$

Quadratic maps as matrices

$$p_s = \sum \gamma_{ij}^{(s)} x_i x_j \quad = \quad (x_1, \ldots, x_N) \underbrace{\begin{pmatrix} \gamma_{11} & \cdots & \frac{\gamma_{1N}}{2} \\ & & \\ \frac{\gamma_{N1}}{2} & \cdots & \gamma_{NN} \end{pmatrix}}_{P^{(s)} \in \mathcal{M}_{N\times N}(\mathbb{F}_q)} \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}$$

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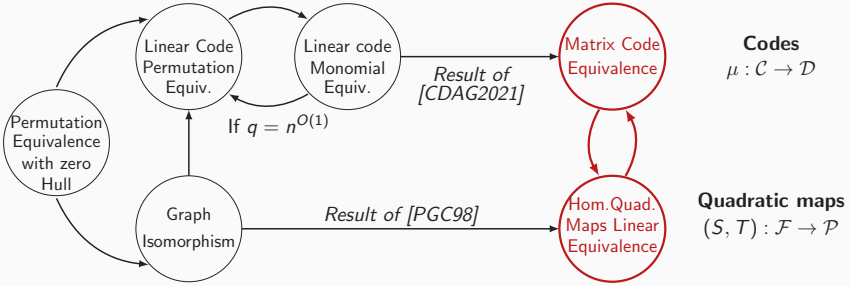
so $\mathcal{P} = (p_1, \ldots, p_k)$ can be seen as matrices $P^{(1)}, \ldots, P^{(k)}$ spanning some code $\mathcal{D}$

How hard is MCE?

Remember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If $q = n^{O(1)}$
Result of [CDAG2021]
Result of [PGC98]
Codes
$\mu : \mathcal{C} \to \mathcal{D}$
Quadratic maps
$(S, T) : \mathcal{F} \to \mathcal{P}$

How hard is MCE?

MCE **is at least as hard as** hQMLE

Proof *(Sketch)*.

- Given $\mathcal{F}$ and $\mathcal{P}$ such that $\mathcal{P}(\mathrm{x}) = \mathcal{F}(\mathrm{xS}) \cdot \mathrm{T}$

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where is T...? Just a change of basis for the code $\mathcal{D}$!

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Proof *(Sketch)*.

- Given codes $\mathcal{C}, \mathcal{D} \subset \mathcal{M}_{m\times n}(\mathbb{F}_q)$ such that $\mu : \mathcal{C} \to \mathcal{D}$ by $\mathsf{C} \mapsto \mathsf{ACB}$

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- so $\tilde{\mu} : \tilde{C} \to \tilde{D}$ gives hQMLE instance with $\mathcal{F}$ and $\mathcal{P}$ in $(n+m)$ variables. □

Solving MCE

Algorithms for QMLE:

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Solving MCE as hQMLE

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Complexity of solving MCE: $\mathcal{O}^*(q^{\frac{2}{3}(n+m)})$ (success prob. $\approx 63\%$)

- map $\mathcal{C}, \mathcal{D}$ to $\tilde{\mathcal{C}}, \tilde{\mathcal{D}}$, gives $N = n + m$ instance of hQMLE
- apply above approach, assuming polynomial-time solver for $\mathcal{P}', \mathcal{F}'$

Solving MCE as hQMLE - taking advantage of bilinearity

- Remark: MCE reduces to **bilinear** hQMLE for $\mathrm{x} = (x_1, \ldots, x_n)$ and $\mathrm{y} = (x_{n+1}, \ldots, x_{n+m})$

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  - Can find collisions (α, β) in the roots!
- However:
  - assuming $n \approx m$, number of roots approximately $q^{n+m-k-1}$
  - so, no non-trivial zeros for $k > n + m$ with reasonable probability

Solving MCE as QMLE - taking advantage of bilinearity

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- Turn $\mathcal{C}, \mathcal{D}$ into $\mathcal{F}$ and $\mathcal{P}$ as before

Solving MCE as QMLE - taking advantage of bilinearity

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- Calculate $\ker \mathsf{F}_\mathsf{b}$ for all $\mathsf{b} \in \mathbb{F}_q^m$
  - if non-trivial, store b with $\ker \mathsf{F}_\mathsf{b}$ (usually 1-dimensional)
- Do the same for $\mathcal{P}$, and feed possible zero-pairs of $\mathcal{F}$ and $\mathcal{P}$ to solver

Complexity of solving MCE with roots: $\mathcal{O}^*(q^m)$

- MCE has roots when $k \leqslant n + m$ with $n = m$
- notice: 100% success instead of $\approx 63\%$
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TO DO / exercise for the listener:

- Can we use bilineairity of MCE when $k > n + m$?
- Can we use zeros in birthday-based approach?

Experiments

Runtimes of Algorithm 1.

$m = n$	k	Runtime (s) SampleSet	Runtime (s) CollisionFind	**Runtime (s) total**	Success probability
11	22	47	180	**227**	0.79
12	24	76	903	**979**	0.71

Runtimes of Algorithm 2.

$m = n$	k	Runtime (s) SampleZeros	Runtime (s) CollisionFind	**Runtime (s) total**	% instances with roots
11	22	0.15	104	**104**	59%
	20	0.16	93	**93**	99%
12	24	0.32	230	**230**	63%
	22	0.29	235	**235**	99%

Thank you for listening!